Fun with Integrals

Consider the integral

$$\iint\limits_R \frac{\alpha(1-2\alpha)k^2 - \gamma(1-2\gamma){k'}^2}{m^2 + (\alpha-\alpha^2)k^2 + (\gamma-\gamma^2){k'}^2 + 2\alpha\gamma k k'}\, d\alpha\, d\gamma,$$

where R is the region in the α-γ plane bounded by the lines $\alpha = 0$, $\gamma = 0$ and $\alpha + \gamma = 1$, and k and k' are positive real constants (although only the constancy of the ks matter, as long as the denominator in the above integral is nonzero).

A suggestive method is to try to use Green's theorem in the plane, specifically

$$\iint\limits_R \left(\frac{\partial Q}{\partial \gamma} - \frac{\partial P}{\partial \alpha}\right) d\alpha\, d\gamma = \oint_C P\, d\alpha + Q\, d\gamma,$$

where C is the contour that bounds R. So, we seek $P(\alpha, \gamma)$ and $Q(\alpha, \gamma)$ so that the above integrand is $\frac{\partial Q}{\partial \gamma} - \frac{\partial P}{\partial \alpha}$.

Well, almost. We need to write the above integral as

$$\begin{aligned}
&\iint\limits_R \Big(\Big)\, d\alpha\, d\gamma \\
&\quad = \iint\limits_R \frac{\alpha(1-2\alpha)k^2 + 2\gamma k k' - \gamma(1-2\gamma){k'}^2 - 2\alpha k k'}{m^2 + (\alpha-\alpha^2)k^2 + (\gamma-\gamma^2){k'}^2 + 2\alpha\gamma k k'}\, d\alpha\, d\gamma \\
&\qquad + \iint\limits_R 2kk' \frac{\alpha - \gamma}{m^2 + (\alpha-\alpha^2)k^2 + (\gamma-\gamma^2){k'}^2 + 2\alpha\gamma k k'}\, d\alpha\, d\gamma \\
&\quad = \iint\limits_R \left(\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \gamma}\right) \ln\left[m^2 + (\alpha-\alpha^2)k^2 + (\gamma-\gamma^2){k'}^2 + 2\alpha\gamma k k'\right] d\alpha\, d\gamma \\
&\qquad + \iint\limits_R 2kk' \frac{\alpha - \gamma}{m^2 + (\alpha-\alpha^2)k^2 + (\gamma-\gamma^2){k'}^2 + 2\alpha\gamma k k'}\, d\alpha\, d\gamma \\
&\quad = \oint_C \ln\left[m^2 + (\alpha-\alpha^2)k^2 + (\gamma-\gamma^2){k'}^2 + 2\alpha\gamma k k'\right](d\alpha + d\gamma) \\
&\qquad + \iint\limits_R 2kk' \frac{\alpha - \gamma}{m^2 + (\alpha-\alpha^2)k^2 + (\gamma-\gamma^2){k'}^2 + 2\alpha\gamma k k'}\, d\alpha\, d\gamma.
\end{aligned}$$

Well, that's it. (That's it?) Yes, that's it. In the contour integral, $d\alpha + d\gamma = 0$ along the line $\alpha + \gamma = 1$. The endpoints of the parts of the contours along the axes are at points where the argument of the logarithm is m^2, and so those integrals vanish. The remaining area integral can be shown to vanish by symmetry (left as an exercise, which we have all done as part of trying this integral by other methods).