In these notes, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{F}$ will represent vector fields, suitably differentiable, and f and g will represent scalar fields, also suitably differentiable.

Vector Algebra Identities

The following two properties may be verified easily by expressing the vectors in cartesian component form, as in $\mathbf{A} = A_x\,\hat{\mathbf{x}} + A_y\,\hat{\mathbf{y}} + A_z\,\hat{\mathbf{z}}$.

Cyclic Property of Scalar Triple Product

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})\,.$$

When expressed in cartesian coordinates, the above might be recognized as a determinant,

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$

Vector Triple Product

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\,(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}\,(\mathbf{A}\cdot\mathbf{B})\,.$$

More on Vector Differential Operators

It is easily shown (that means you do it) that

$$\begin{aligned}
\boldsymbol{\nabla}(f\,g) &= (\boldsymbol{\nabla} f)\,g + f\,(\boldsymbol{\nabla} g)\\
\boldsymbol{\nabla}\cdot(f\,\mathbf{A}) &= (\boldsymbol{\nabla} f)\cdot\mathbf{A} + f\,(\boldsymbol{\nabla}\cdot\mathbf{A})\\
\boldsymbol{\nabla}\times(f\,\mathbf{A}) &= (\boldsymbol{\nabla} f)\times\mathbf{A} + f\,(\boldsymbol{\nabla}\times\mathbf{A})\\
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) &= 0\\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla} f) &= \mathbf{0}.
\end{aligned}$$

What takes a bit of doing is to show that

$$\boldsymbol{\nabla}(\mathbf{A}\times\mathbf{B}) = (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{B} - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})\,,$$

and we really will need this when we get to energy propagation.

It turns out that the items

$$\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) \qquad \text{and} \qquad \boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})$$

do not have simple forms as extensions of a product rule. Fortunately, we won't need these forms except in special cases, which we will handle, well, specially.

For propagation of waves, we will need the second-order derivative

$$\nabla \times (\nabla \times \mathbf{A}),$$

and other texts give this in a form which I dislike. So, this will also be done on a case-by-case basis. In cartesian coordinates, the above becomes

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 A_x \,\hat{\mathbf{x}} + \nabla^2 A_y \,\hat{\mathbf{y}} + \nabla^2 A_z \,\hat{\mathbf{z}},$$

but this form is *only valid for cartesian coordinates*.

Slightly related to the above is an important point that is often overlooked. Specifically, we have quite immediately,

$$\nabla \cdot \hat{\mathbf{x}} = \nabla \cdot \hat{\mathbf{y}} = \nabla \cdot \hat{\mathbf{z}} = 0,$$

$$\nabla \times \hat{\mathbf{x}} = \nabla \times \hat{\mathbf{y}} = \nabla \times \hat{\mathbf{z}} = \mathbf{0},$$

but in for instance spherical coordinates (you can show this!)

$$\nabla \cdot \hat{\mathbf{r}} = \frac{2}{r}, \qquad \nabla \cdot \hat{\boldsymbol{\theta}} = \frac{\cot\theta}{r}, \qquad \nabla \cdot \hat{\boldsymbol{\phi}} = 0,$$

and slightly more complicated forms for the curls of the unit vectors (although we'll be using $\nabla \times \hat{\mathbf{r}} = \mathbf{0}$ almost right away).

An Operative Definition of the Gradient

A rigorous definition is more than we need or probably want. Here's a useful definition that serves our purposes:

> If a scalar field f is sufficiently differentiable that there exists a linear relation between the infinitesimal change df in the field and the change $d\mathbf{r}$ in the independent variables, then there exisits a vector field ∇f, the "gradient of f," such that
>
> $$df = (\nabla f) \cdot d\mathbf{r}.$$

Let's use this to find the expression for the gradient in spherical coordinates. Start with a *purely mathematical exposition*,

$$df = \frac{\partial f}{\partial r}\, dr + \frac{\partial f}{\partial \theta}\, d\theta + \frac{\partial f}{\partial \phi}\, d\phi.$$

This is "purely mathematical" because the mathematical definition of partial derivatives is formulated without respect to any geometric interpretation of the variables.

However, we have, in spherical coordinates,

$$d\mathbf{r} = dr\,\hat{\mathbf{r}} + r\,\hat{\boldsymbol{\theta}} + r\,\sin\theta\,\hat{\boldsymbol{\phi}};$$

this relation establishes the geometric relation between the coordinates.

Equating the expressions for df,

$$(\boldsymbol{\nabla} f)\cdot\left(dr\,\hat{\mathbf{r}} + r\,\hat{\boldsymbol{\theta}} + r\,\sin\theta\,\hat{\boldsymbol{\phi}}\right) = \frac{\partial f}{\partial r}\,dr + \frac{\partial f}{\partial \theta}\,d\theta + \frac{\partial f}{\partial \phi}\,d\phi.$$

Equating the terms multiplying each differential element (we can do that if the coordinates are *independent*, which they are),

$$(\boldsymbol{\nabla} f) = \frac{\partial f}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\,\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}},$$

as seen elsewhere. That is, $\boldsymbol{\nabla} f$ is that vector which must be dotted into $d\mathbf{r}$ to produce df.

A Synopsis of the Fundamental Theorems of Vector Calculus

From the above exposition of the gradient, $df = (\boldsymbol{\nabla} f)\cdot d\mathbf{r}$, we can immediately cite the SWOP

$$\int_{\mathbf{a}}^{\mathbf{b}} (\boldsymbol{\nabla} f)\cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}).$$

An alternate phrasing is

$$\text{If}\qquad \mathbf{F} = \boldsymbol{\nabla} f,\qquad \text{then}\qquad \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F}\cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}).$$

From this we see that if $\mathbf{b} = \mathbf{a}$, the line integral is along a closed path, and the integral is zero. That is, if $\mathbf{F}$ is the gradient of a scalar function, adding up the increments $df = \mathbf{F}\cdot d\mathbf{r}$ gets you back where you started, and no net change in the value of f.

However, not every vector field may be written as the gradient of a scalar (try to write $y\,\hat{\mathbf{x}} - x\,\hat{\mathbf{y}}$ as a gradient). In this situation, we would have

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot d\mathbf{A},$$

where in the first integral, the contour C is the boundary of the surface S, assuming both integrals are defined (and some geometrical constraints on the smoothness of the surface. When we need this, and we will, we'll emphasize that there are an infinite number of surfaces bounded by a given closed contour, and the above result is valid for any such surface.

Lastly, if in the above the contour "shrinks" to zero, the surface has no boundary and is a closed surface, and so the surface integral of the flux of the curl over a close surface is zero. However, if the integrand is not the curl of another vector function, the integral in general will not vaninsh, and we would have

$$\oiint_S \mathbf{G} \cdot d\mathbf{a} = \iiint_V \boldsymbol{\nabla} \cdot \mathbf{G} \, dV$$

where the closed surface S is the boundary of the volume V.