8.022 at ESG **Fall 2004**

Problem Set 01 **due Friday September 17**

Reading

Chapter 1 (Purcell), Sections 1.1-1.7, Appendix E, endpapers.

Problems from the text

- **1.1**
- **1.3**
- **1.4**
- **1.8**
- **1.9**
- **1.10**
- **1.12**
- **1.15**
- **1.16** (a classic, and one which we'll see in a similar form for magnetism)

Math Exercises

These are indeed exercises, in that you won't be finding any new results, but you'll want to convince yourself (and the graders) that you're comfortable with manipulations of partial derivatives. You will need to consult your table of **Vector Operators in Cylindrical and Spherical Coordinates** or a similar table.

If you're very familiar with these manipulations, you might find these slightly tedious; it's entirely possible you've done some of these before. If not, it's good to do such problems at least once, but you may not want to do them more than once.

At the end of this problem set is a statement (not a proof) of a special case of the **Chain Rule for Partial Differentiation**.

(A) (This is meant to be quite easy.) For each of the three coordinate systems, show by combination of the expressions for $\boldsymbol{\nabla}\psi$ and $\boldsymbol{\nabla}\cdot\mathbf{A}$ leads to the expression for $\nabla^2\psi = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\psi)$.

Verify that the statement at the end of the Spherical Coordinates page,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\,\psi)\,,$$

is correct.

(B) Define scalar functions in spherical coordinates by

$$\varphi_1 = r\,\cos\theta, \qquad \varphi_{-2} = \frac{\cos\theta}{r^2}, \qquad r > 0.$$

Find $\boldsymbol{\nabla}\varphi_1$ and $\boldsymbol{\nabla}\varphi_{-2}$ and show that $\nabla^2\varphi_1 = \nabla^2\varphi_{-2} = 0$.

Now, express φ_1 and φ_{-2} in cartesian coordinates and repeat the above calculations. Then (you guessed it) convert to cylindrical coordinates and repeat the above calculations.

(C) Suppose we have a scalar function in one coordinate system and we wish to change coordinates; if you've done problem **(B)** you might appreciate why we might want to do this. Specifically, suppose we are given $f = f(x,\, y,\, z)$ and we have

$$f(x,\, y,\, z) = g(r,\, \theta,\, z),$$

with $g = g(r,\, \theta\, z)$ in cylindrical coordinates. Using the explicit forms for $\nabla^2 f$ and $\nabla^2 g$ as given in the tables, show that $\nabla^2 f = \nabla^2 g$. That is, express the second partials of f in cartesian coordinates in terms of the partials of g in cylindrical coordinates, using the **Chain Rule for Partial Derivatives**, given (but not derived) in part on the next page. Since you'll be wanting expressions for second derivatives, you'll need to use the Chain Rule twice.

If you had infinite time, you could try the same for conversion to spherical coordinates. Save this for a very long plane ride.

Chain Rule for Partial Derivatives

(a partial explanation)

I'll use η and ζ to avoid confusion with any other symbols used so far for scalar functions. When spoken, these sound similar. Basically, use whatever symbols you wish.

Suppose

$$\eta(x,\, y,\, z) = \zeta(u,\, v,\, w),$$

where η and ζ are sufficiently differentiable scalar functions and u, v, w are sufficiently differentiable functions of x, y and z. We would like to find the partial derivatives of η in terms of ζ, u, v and w. Our motivation is this: If, for instance x changes, then in general u, v and w will change, and so we need to account for how ζ changes with respect to each of u, v and w.

So, here's a SWOP (Statement WithOut Proof – we'll do a lot of these):

$$\frac{\partial \eta}{\partial x} = \frac{\partial \zeta}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \zeta}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial \zeta}{\partial w}\frac{\partial w}{\partial x},$$

with of course the similar statements for $\frac{\partial \eta}{\partial y}$ and $\frac{\partial \eta}{\partial z}$.

We can and will be slick, as long as the slickness doesn't obscure the physics. The above statements can be combined into a single matrix equation,

$$\begin{bmatrix} \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \eta}{\partial y} \\ \dfrac{\partial \eta}{\partial z} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} & \dfrac{\partial w}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} & \dfrac{\partial w}{\partial y} \\ \dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial z} & \dfrac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \zeta}{\partial u} \\ \dfrac{\partial \zeta}{\partial v} \\ \dfrac{\partial \zeta}{\partial w} \end{bmatrix}.$$

It's expressions such as this that beg use of the subscript notation for partial derivatives, as in $\frac{\partial f}{\partial x} = f_x$, so that the above would be

$$\begin{bmatrix} \eta_x \\ \eta_y \\ \eta_z \end{bmatrix} = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & v_y \\ u_z & v_z & w_z \end{bmatrix} \begin{bmatrix} \zeta_u \\ \zeta_v \\ \zeta_w \end{bmatrix}.$$

Trust me, the second form is much easier to typeset.

There are even more compact notations, and if we have more dimensions, other notations would come in handy.

One drawback is that in Purcell's notation, f_x would be the x-component of the vector $\mathbf{f}$. So, we will use the subscript notation only when there is little chance of ambiguity, and we'll always state so explicitly when we're using this notation.

The bottom line is, it's up to you to make sure the notation we use agrees with whatever notation you use in any other math or physics subjects.