

## II. Coulomb's Law – Worked Examples

**Example 1: Charge conservation**

**Example 2: Electric force in hydrogen atom**

**Example 3: Compare electrical and gravitational force**

**Example 4: Superposition principle for electric forces**

**Example 5: Electric field of a proton**

**Example 6: Charged particle moving through charged plates**

**Example 7: Millikan oil drop**

**Example 8: Charge moving perpendicular to an Electric field**

**Example 9: Electric field of a dipole ion Cartesian coordinates**

**Example 10: Electric dipole field in polar coordinates**

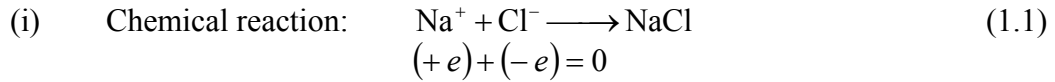
**Example 11: Electric field due to a uniformly charged wire**

**Example 12: Electric field due to a uniformly charged disk**

**Example 13: Electric field of two parallel discs**

### Example 1. Charge conservation

In an isolated system, the total charges stay constant, i.e. in a closed system, charge can neither be created nor destroyed. It is transferred from one body to another. One can readily show that charge is conserved in the following processes



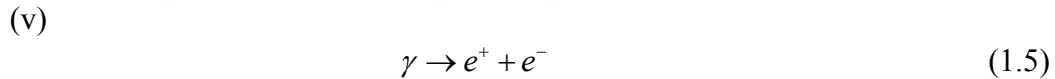
(ii) Radioactive decay: A neutron will decay into a positively charged proton, a negatively charged electron and a neutral charged electron-anti-neutrino,



(iii) Annihilation: electron-positron annihilation



(iv) Pair production: electron-positron production



where a  $\gamma$ -ray is a high energy charge-neutral particle of light.

**Example 2:** The electrical force between a proton and an electron in a hydrogen atom.

**Question:** In the classical model of the hydrogen atom, the electron revolves around the proton with a radius of  $r = 0.53 \times 10^{-10}$  m. What is the magnitude of the electric force between the proton and the electron? The charge of the electron and proton have magnitude  $e = 1.6 \times 10^{-19}$  C.

**Answer:** The magnitude of the force is given by

$$F_e = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2}$$
(2.1)

**Principle for Calculating Numerical Results from Symbolic Expressions:** When calculating numerical quantities, first determine a symbolic expression for the desired quantity and then substitute the numerical values into the symbolic expression. When you

have finished the calculation, make sure your answer agrees with the proper amount of significant digits. You should report your answer with the same number of significant digits as the given quantity that has the least number of significant digits.

Now we can substitute our numerical values and find that the magnitude of the force between the proton and the electron in the hydrogen atom is

$$F_e = \frac{(9.0 \times 10^9 \text{ N} \cdot \text{m}^2 / \text{C}^2)(1.6 \times 10^{-19} \text{ C})^2}{(5.3 \times 10^{-11} \text{ m})^2} = 8.2 \times 10^{-8} \text{ N} \quad (2.2)$$

**Example 3:** Comparing the electrical and gravitational force between electron and proton.

**Question:** What is ratio of the magnitudes of the electrical and gravitational force between electron and proton if we assume that the electron and proton are separated by a distance equal to the classical electron orbit in the hydrogen atom,  $r = 0.53 \times 10^{-10} \text{ m}$ ? The mass of the electron is  $m_e = 9.1 \times 10^{-31} \text{ kg}$  and the mass of the proton is  $m_p = 1.7 \times 10^{-27} \text{ kg}$ . Does this ratio depend on the distance between the proton and the electron? In light of your calculation, explain why electrical forces do not influence the motion of planets.

**Answer:** The ratio of the magnitudes of the electrical and gravitational force is given by

$$\gamma = \frac{\left( \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} \right)}{\left( G \frac{m_p m_e}{r^2} \right)} = \frac{\frac{1}{4\pi\epsilon_0} e^2}{G m_p m_e} = \frac{(9.0 \times 10^9 \text{ N} \cdot \text{m}^2 / \text{C}^2)(1.6 \times 10^{-19} \text{ C})^2}{(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 / \text{kg}^2)(1.7 \times 10^{-27} \text{ kg})(9.1 \times 10^{-31} \text{ kg})} = 2.2 \times 10^{39} \quad (3.1)$$

which is independent of the distance between the proton and the electron.

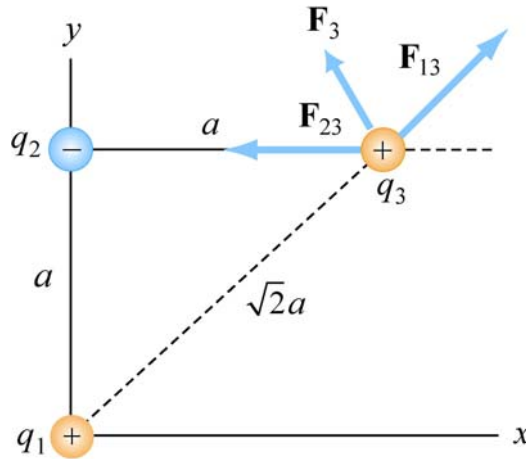
The electric force is 39 orders of magnitude stronger than the gravitational force between the electron and the proton. Then why are the large scale motions of planets determined by the gravitational force and not the electrical force. The answer is that the magnitudes of the charge of the electron and proton are equal. The best experiments show that the difference between these magnitudes is a number on the order of  $10^{-24}$ . Since objects like planets have about the same number of protons as electrons, they are essentially electrically neutral. Therefore the force between planets is entirely determined by gravity.

**Example 4: Superposition Principle for Electric forces**

**Question:** In Figure 4.1 shown below, three charges are interacting. Find the force on the charge  $q_3$  assuming that  $q_1 = 6.0 \times 10^{-6} \text{ C}$ ,  $q_2 = -q_1 = -6.0 \times 10^{-6} \text{ C}$ ,  $q_3 = +3.0 \times 10^{-6} \text{ C}$  and  $a = 2.0 \times 10^{-2} \text{ m}$ .

**Answer:** Using the superposition principle, we have

$$\vec{\mathbf{F}}_3 = \vec{\mathbf{F}}_{13} + \vec{\mathbf{F}}_{23} = \frac{1}{4\pi\epsilon_0} \left( \frac{q_1 q_3}{r_{13}^2} \hat{\mathbf{r}}_{13} + \frac{q_2 q_3}{r_{23}^2} \hat{\mathbf{r}}_{23} \right) \quad (4.1)$$



**Figure 4.1** Illustration of the superposition principle.

Since the unit vectors  $\hat{\mathbf{r}}_{13}$  and  $\hat{\mathbf{r}}_{23}$  do not point in the same directions, in order to mathematically compute the vector sum, we must express each unit vectors in terms of their Cartesian components and add the forces according to the principles of vector addition.

Recall that  $\hat{\mathbf{r}}_{13}$  is located at charge  $q_3$  and points from  $q_1$  to  $q_3$ . From Figure 4.1, we see that

$$\hat{\mathbf{r}}_{13} = \cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}} = \frac{\sqrt{2}}{2} (\hat{\mathbf{i}} + \hat{\mathbf{j}}) \quad (4.2)$$

The unit vector  $\hat{\mathbf{r}}_{23} = \hat{\mathbf{i}}$  points from  $q_2$  to  $q_3$ . Therefore, the total force is

$$\vec{\mathbf{F}}_3 = \frac{1}{4\pi\epsilon_0} \left( \frac{q_1 q_3}{r_{13}^2} \hat{\mathbf{r}}_{13} + \frac{q_2 q_3}{r_{23}^2} \hat{\mathbf{r}}_{23} \right) = \frac{1}{4\pi\epsilon_0} \left( \frac{q_1 q_3}{(\sqrt{2}a)^2} \frac{\sqrt{2}}{2} (\hat{\mathbf{i}} + \hat{\mathbf{j}}) + \frac{(-q_1) q_3}{a^2} \hat{\mathbf{i}} \right) \quad (4.3)$$

Now we add components and find that

$$\vec{F}_3 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_3}{a^2} \left[ \left( \frac{\sqrt{2}}{4} - 1 \right) \hat{i} + \frac{\sqrt{2}}{4} \hat{j} \right] \quad (4.4)$$

The magnitude of the total force is given by

$$\begin{aligned} F_3 &= \frac{1}{4\pi\epsilon_0} \frac{q_1 q_3}{a^2} \left[ \left( \frac{\sqrt{2}}{4} - 1 \right)^2 + \left( \frac{\sqrt{2}}{4} \right)^2 \right]^{1/2} \\ &= (.74) \frac{(9.0 \times 10^9 \text{ N} \cdot \text{m}^2 / \text{C}^2)(6.0 \times 10^{-6} \text{ C})(3.0 \times 10^{-6} \text{ C})}{(2.0 \times 10^{-2} \text{ m})^2} = 3.0 \text{ N} \end{aligned} \quad (4.5)$$

The angle that the force makes with the positive  $x$ -axis is

$$\phi = \tan^{-1} \left( \frac{F_{3,y}}{F_{3,x}} \right) = \tan^{-1} \left[ \frac{\sqrt{2}/4}{-1 + \sqrt{2}/4} \right] = 151.3^\circ \quad (4.6)$$

(Note there are two solutions to this equation. The second solution  $\phi = -28.7^\circ$  is incorrect because it indicates that the force has negative  $\hat{i}$  and  $\hat{j}$  components.)

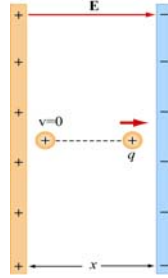
### Example 5: Electric field of a proton

**Question:** A proton has charge  $e = 1.6 \times 10^{-19} \text{ C}$ . What is the magnitude of the electric field a distance  $r = 0.5 \times 10^{-10} \text{ m}$  from the center of the proton? We shall ignore any internal structure of the proton and assume that all the charge is concentrated at the center of the proton. Then the magnitude of the electric field is given by

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} = \frac{(9.0 \times 10^9 \text{ N} \cdot \text{m}^2 / \text{C}^2)(1.6 \times 10^{-19} \text{ C})}{(0.5 \times 10^{-10} \text{ m})^2} = 5.76 \times 10^{11} \text{ N/C} \quad (5.1)$$

### Example 6: Charged Particle moving through charged plates

Consider a charge  $+q$  moving between two parallel plates of opposite charges, as shown in Figure 6.1 below.



**Figure 6.1** Charge moving in a constant electric field

**Question 1:** What is the acceleration on a charged particle in a uniform electric field  $\vec{E} = E_x \hat{i}$  produced by two parallel plates of opposite charges?

**Answer:** From Newton's second law, the charged particle will experience a force

$$\vec{F}_e = q\vec{E} = m\vec{a} \quad (6.1)$$

Since the electric field points in the  $+x$ -direction, the acceleration is

$$\vec{a} = \frac{\vec{F}_e}{m} = \frac{q\vec{E}}{m} = \frac{qE_x}{m} \hat{i} \quad (6.2)$$

**Question 2:** Suppose the charge  $+q$  is released from rest. What is the velocity of the charge after it has traveled a distance  $d$  in the  $+x$ -direction?

**Answer:** The problem is a one-dimensional kinematics problem with constant acceleration. Suppose the particle is at rest ( $v_0 = 0$ ) when it is first released from the positive plate. The terminal speed  $v$  of the particle as it strikes the negative plate is

$$v = \sqrt{2ad} = \sqrt{2dqE_x/m} \quad (6.3)$$

**Question 3:** What is the kinetic energy of the charge after it has traveled a distance  $d$  in the  $+x$ -direction?

**Answer:** The kinetic energy is given by

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(2ad) = qE_x d \quad (6.4)$$

### Example 7: Millikan oil drop

An oil drop of radius  $r = 1.64 \times 10^{-6}$  m and mass density  $\rho_{oil} = 8.51 \times 10^2$  kg/m<sup>3</sup> is allowed to fall from rest a distance  $d_0 = 8.0 \times 10^{-2}$  m and then enters into a region of constant external field  $\vec{E}$  applied in the downward direction. The oil drop has an unknown electric charge  $q$  (due to irradiation by bursts of X-rays). The magnitude of the electric field is adjusted until the gravitational force  $\vec{F}_g = m\vec{g} = -mg\hat{j}$  on the oil drop is exactly ‘balanced’ by the electric force,  $\vec{F}_e = q\vec{E}$ . During this balancing the oil drop has moved downwards an additional distance  $d_1 = 1.2 \times 10^{-1}$  m. This balancing occurs when the  $\vec{E} = E_y\hat{j} = -(1.92 \times 10^5 \text{ N/C})\hat{j}$ . Note that the y-component of the electric field is negative.

**Question 1:** What is the mass of the oil drop?

**Answer:** The mass density  $\rho_{oil}$  times the volume of the oil drop will yield the total mass  $M$  of the oil drop,

$$M = \rho_{oil}V = \rho_{oil}\left(\frac{4}{3}\pi r^3\right) \quad (7.1)$$

where the oil drop is assumed to be a sphere of radius  $r$  with volume  $V = 4\pi r^3 / 3$ .

Now we can substitute our numerical values into our symbolic expression for the mass,

$$M = \rho_{oil}\left(\frac{4}{3}\pi r^3\right) = (8.51 \times 10^2 \text{ kg/m}^3)\left(\frac{4\pi}{3}\right)(1.64 \times 10^{-6} \text{ m})^3 = 1.57 \times 10^{-14} \text{ kg} \quad (7.2)$$

**Question 2:** What is the charge on the oil drop in units of electronic charge  $e = 1.6 \times 10^{-19}$  C?

**Answer:** The oil drop will be in static equilibrium when the gravitational force exactly balances the electrical force,

$$\vec{F}_g + \vec{F}_e = \vec{0} \quad (7.3)$$

Using our force laws, we have

$$\begin{aligned} 0 &= m\vec{g} + q\vec{E} \\ &= -mg\hat{j} + qE_y\hat{j} \end{aligned} \quad (7.4)$$

The gravitational force points downward, so therefore the force on the oil must be upward. Since the electrical field points downward, the charge on the oil drop must be negative.

Notice that we have chosen the unit vector  $\hat{j}$  to point upward. We can solve this equation for the charge on the oil drop:

$$q = \frac{mg}{E_y} = \frac{(1.57 \times 10^{-14} \text{ kg})(9.80 \text{ m/s}^2)}{-1.92 \times 10^5 \text{ N/C}} = -8.03 \times 10^{-19} \text{ C} \quad (7.5)$$

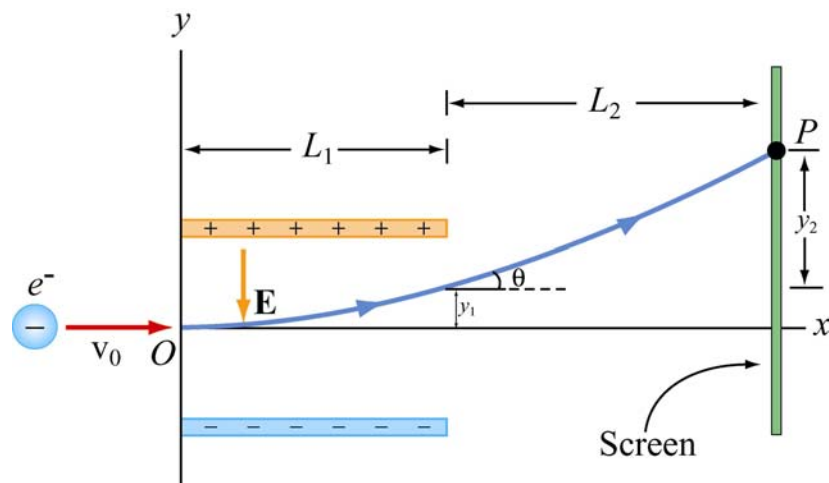
Since the electron has charge  $e = 1.6 \times 10^{-19} \text{ C}$ , the charge of the oil drop in units of  $e$  is

$$N = \frac{q}{e} = \frac{8.02 \times 10^{-19} \text{ C}}{1.6 \times 10^{-19} \text{ C}} = 5 \quad (7.6)$$

You may at first be surprised that this number is an integer, but the Millikan Oil Drop experiment was the first direct experimental evidence that charge is quantized. Thus from the given data we can assert that there are five electrons on the oil drop!

### Example 8: Charge Moving Perpendicular to an Electric Field

Let us now consider what happens when an electron is injected horizontally into a uniform field produced by two oppositely charged plates that points downward.



**Figure 8.1** Charge moving perpendicular to an Electric field

The particle has an initial velocity  $\vec{v}_0 = v_0 \hat{i}$  perpendicular to the electric field  $\vec{E} = E_y \hat{j}$  where  $E_y < 0$ . The motion of the electron is analogous to the motion of a mass that is thrown horizontally in a constant gravitational field. The mass follows a parabolic



trajectory downward. Since the electron is negatively charged, the constant force on the electron is upward and the electron will be deflected upwards on a parabolic path.

**Question 1:** What is the force on the electron?

**Answer:** Since the electron has a negative charge,  $q = -e$ , the force on the electron is

$$\vec{\mathbf{F}}_e = q\vec{\mathbf{E}} = -e\vec{\mathbf{E}} = -eE_y\hat{\mathbf{j}} \quad (8.1)$$

Since  $E_y < 0$ , the force on the electron is upward.

**Question 2:** What is the acceleration of the electron?

**Answer:** The acceleration of the electron is

$$\vec{\mathbf{a}} = \frac{q\vec{\mathbf{E}}}{m} = \frac{qE_y}{m}\hat{\mathbf{j}} = -\frac{eE_y}{m}\hat{\mathbf{j}} \quad (8.2)$$

The acceleration of the electron is also upward.

**Question 3:** The plates have length  $L_1$  in the  $x$ -direction. At what time  $t_1$  will the electron to leave the plate?

**Answer:** The time of passage for the electron is given by  $t_1 = L_1/v_0$ .

**Question 4:** Suppose the electron enters the electric field at time  $t = 0$ . What is the velocity of the electron at time  $t_1$  when it leaves the plates?

**Answer:** The electron has an initial horizontal velocity,  $\vec{\mathbf{v}}_0 = v_0\hat{\mathbf{i}}$ . Since the acceleration of the electron is in the  $+y$ -direction, only the  $y$ -component of the velocity changes. The velocity at a later time  $t_1$  is given by

$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} = v_0\hat{\mathbf{i}} + a_y t_1\hat{\mathbf{j}} = v_0\hat{\mathbf{i}} + \left(-\frac{eE_y}{m}\right)t_1\hat{\mathbf{j}} \quad (8.3)$$

**Question 5:** What is the vertical displacement of the electron after time  $t_1$  when it leaves the plates?

**Answer:** From the figure, we see that the electron travels a horizontal distance  $L_1$  in the time  $t_1 = L_1/v_0$  and then emerges from the plates with a vertical displacement

$$y_1 = \frac{1}{2} a_y t_1^2 = -\frac{1}{2} \left( \frac{eE_y}{m} \right) \left( \frac{L_1}{v_0} \right)^2 \quad (8.4)$$

**Question 6:** What angle  $\theta_1$  does the electron make  $\theta_1$  with the horizontal, when the electron leaves the plates at time  $t_1$ ?

**Answer:** When the electron leaves the plates at time  $t_1$ , the electron makes an angle  $\theta_1$  with the horizontal given by the ratio of the components of its velocity,

$$\tan \theta = \frac{v_y}{v_x} = \frac{(-eE_y/m)(L_1/v_0)}{v_0} = -\frac{eE_y L_1}{mv_0^2} \quad (8.5)$$

**Question 7:** The electron hits the screen located a distance  $L_2$  from the end of the plates at a time  $t_2$ . What is the total vertical displacement of the electron from time  $t = 0$  until it hits the screen at time  $t_2$ ?

**Answer:** After the electron leaves the plate, there is no longer any force on the electron so it travels in a straight path. The deflection  $y_2$  is

$$y_2 = L_2 \tan \theta_1 = -\frac{eE_y L_1 L_2}{mv_0^2} \quad (8.6)$$

and the total deflection becomes

$$y = y_1 + y_2 = -\frac{1}{2} \frac{eE_y L_1^2}{mv_0^2} - \frac{eE_y L_1 L_2}{mv_0^2} = -\frac{eE_y L_1}{mv_0^2} \left( \frac{1}{2} L_1 + L_2 \right) \quad (8.7)$$

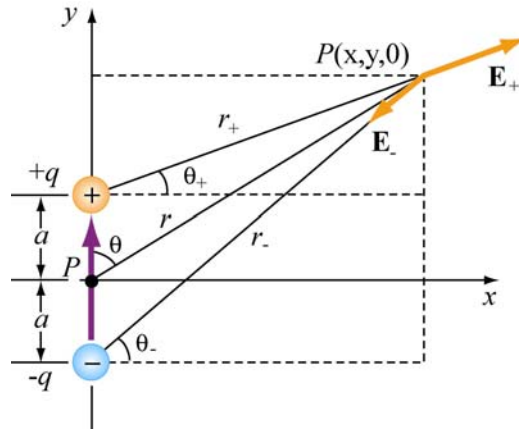
### Example 9: (challenging) Electric field of a dipole ion Cartesian Coordinates

Show that the electric field of the dipole in the limit where  $r \gg a$  is

$$E_x = \frac{3p}{4\pi\epsilon_0 r^3} \sin\theta \cos\theta \quad (9.1)$$

$$E_y = \frac{p}{4\pi\epsilon_0 r^3} (3\cos^2\theta - 1) \quad (9.2)$$

where  $\sin\theta = x/r$  and  $\cos\theta = y/r$ .



**Figure 9.1** Electric field at point  $P$

**Solution:** Let's compute the electric field strength at a distance  $r \gg a$  due to the dipole. The  $x$ -component of the electric field strength at the point  $P$  with Cartesian coordinates  $(x, y, 0)$  is given by

$$E_x = \frac{q}{4\pi\epsilon_0} \left( \frac{\cos\theta_+}{r_+^2} - \frac{\cos\theta_-}{r_-^2} \right) = \frac{q}{4\pi\epsilon_0} \left( \frac{x}{[x^2 + (y-a)^2]^{3/2}} - \frac{x}{[x^2 + (y+a)^2]^{3/2}} \right) \quad (9.3)$$

where

$$r_{\pm}^2 = r^2 + a^2 \mp 2ra \cos\theta = x^2 + (y \mp a)^2 \quad (9.4)$$

Similarly, the  $y$ -component reads

$$E_y = \frac{q}{4\pi\epsilon_0} \left( \frac{\sin\theta_+}{r_+^2} - \frac{\sin\theta_-}{r_-^2} \right) = \frac{q}{4\pi\epsilon_0} \left( \frac{y-a}{[x^2 + (y-a)^2]^{3/2}} - \frac{y+a}{[x^2 + (y+a)^2]^{3/2}} \right) \quad (9.5)$$

We shall make a polynomial expansion for the electric field using the binomial theorem. We will then collect terms that are proportional to  $1/r^3$  and ignore terms that are proportional to  $1/r^5$ , where  $r = +(x^2 + y^2)^{1/2}$ .

We begin with

$$[x^2 + (y \pm a)^2]^{-3/2} = [x^2 + y^2 + a^2 \pm 2ay]^{-3/2} = r^{-3} \left[ 1 + \frac{a^2 \pm 2ay}{r^2} \right]^{-3/2} \quad (9.6)$$

In the limit where  $r \gg a$ , we can use the binomial theorem with  $s \equiv \frac{a^2 \pm 2ay}{r^2}$ ,

$$(1 + s)^{-3/2} = 1 - \frac{3}{2}s + \frac{15}{8}s^2 - \dots \quad (9.7)$$

and the above equations for the components of the electric field becomes

$$E_x = \frac{q}{4\pi\epsilon_0} \frac{6xya}{r^5} + \dots \quad (9.8)$$

and

$$E_y = \frac{q}{4\pi\epsilon_0} \left( -\frac{2a}{r^3} + \frac{6y^2a}{r^5} \right) + \dots \quad (9.9)$$

where we have neglected the  $O(a^3)$  terms. The electric field then is

$$\vec{\mathbf{E}} = E_x \hat{\mathbf{i}} + E_y \hat{\mathbf{j}} = \frac{q}{4\pi\epsilon_0} \left[ -\frac{2a}{r^3} \hat{\mathbf{j}} + \frac{6ya}{r^5} (x \hat{\mathbf{i}} + y \hat{\mathbf{j}}) \right] = \frac{p}{4\pi\epsilon_0 r^3} \left[ \frac{3yx}{r^2} \hat{\mathbf{i}} + \left( \frac{3y^2}{r^2} - 1 \right) \hat{\mathbf{j}} \right] \quad (9.10)$$

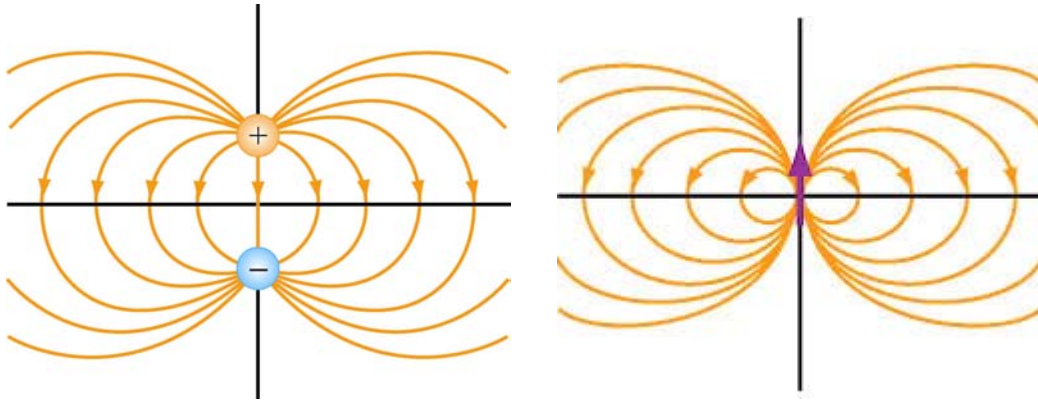
where we have made used of the definition of the magnitude of the electric dipole moment  $p = 2aq$ .

The expression for  $\vec{\mathbf{E}}$  can also be written in terms of the polar coordinates. With  $\sin \theta = x/r$  and  $\cos \theta = y/r$  (as seen from Figure 9.1), we have

$$E_x = \frac{3p}{4\pi\epsilon_0 r^3} \sin \theta \cos \theta \quad (9.11)$$

and

$$E_y = \frac{p}{4\pi\epsilon_0 r^3} (3 \cos^2 \theta - 1) \quad (9.12)$$



**Figure 9.2** Field lines for (a) a finite dipole, and (b) a point dipole

**Example 10: (challenging) Electric dipole field in polar coordinates**

Show that the expression for the electric field of a dipole,  $\vec{\mathbf{E}}$ , can also be written in terms of the polar coordinates as

$$\vec{\mathbf{E}}(r, \theta) = E_r \hat{\mathbf{r}} + E_\theta \hat{\boldsymbol{\theta}} \quad (10.1)$$

where

$$E_r = \frac{2p \cos \theta}{4\pi\epsilon_0 r^3} \quad (10.2)$$

and

$$E_\theta = \frac{p \sin \theta}{4\pi\epsilon_0 r^3} \quad (10.3)$$

**Solution:** We begin with our expression for the electric dipole in Cartesian coordinates,

$$E_x = \frac{3p}{4\pi\epsilon_0 r^3} \sin \theta \cos \theta \quad (10.4)$$

$$E_y = \frac{p}{4\pi\epsilon_0 r^3} (3 \cos^2 \theta - 1) \quad (10.5)$$

Then

$$\vec{\mathbf{E}}(r, \theta) = \frac{p}{4\pi\epsilon_0 r^3} \left[ 3 \sin \theta \cos \theta \hat{\mathbf{i}} + (3 \cos^2 \theta - 1) \hat{\mathbf{j}} \right] \quad (10.6)$$

Using a little algebra,

$$\vec{\mathbf{E}}(r, \theta) = \frac{P}{4\pi\epsilon_0 r^3} \left[ 2 \cos \theta (\sin \theta \hat{\mathbf{i}} + \cos \theta \hat{\mathbf{j}}) + \sin \theta \cos \theta \hat{\mathbf{i}} + (\cos^2 \theta - 1) \hat{\mathbf{j}} \right] \quad (10.7)$$

Further use of the trigonometric identity  $(\cos^2 \theta - 1) = -\sin^2 \theta$  yields

$$\vec{\mathbf{E}}(r, \theta) = \frac{P}{4\pi\epsilon_0 r^3} \left[ 2 \cos \theta (\sin \theta \hat{\mathbf{i}} + \cos \theta \hat{\mathbf{j}}) + \sin \theta (\cos \theta \hat{\mathbf{i}} - \sin \theta \hat{\mathbf{j}}) \right] \quad (10.8)$$

The unit vectors in polar coordinates have Cartesian decomposition becomes

$$\hat{\mathbf{r}} = \sin \theta \hat{\mathbf{i}} + \cos \theta \hat{\mathbf{j}} \text{ and } \hat{\boldsymbol{\theta}} = \cos \theta \hat{\mathbf{i}} - \sin \theta \hat{\mathbf{j}} \quad (10.9)$$

Thus the electric field in polar coordinates is given by

$$\boxed{\vec{\mathbf{E}}(r, \theta) = \frac{P}{4\pi\epsilon_0 r^3} \left[ 2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}} \right]} \quad (10.10)$$

and the magnitude of  $\vec{\mathbf{E}}$  is given by

$$E = (E_r^2 + E_\theta^2)^{1/2} = \frac{P}{4\pi\epsilon_0 r^3} (3 \cos^2 \theta + 1)^{1/2} \quad (10.11)$$

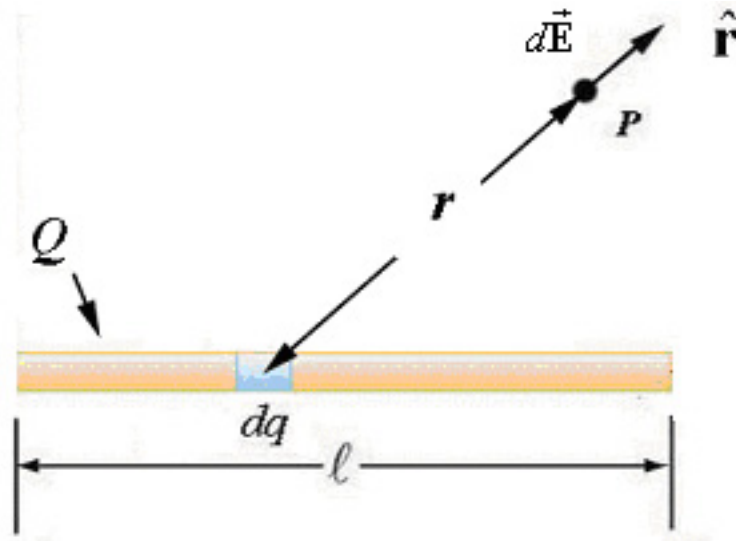
### Example 11: Electric field due to a Uniformly Charged Wire

Let's now calculate the electric field of the uniformly charged wire of length  $L$  is uniformly charged with total charge  $Q$ . The uniform linear charge density is  $\lambda = Q/L$ . The electric field everywhere in space can be found by considering a small charge element  $dq$  located on the wire. The electric field at a point  $P$  is given by Coulomb's Law,

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \hat{\mathbf{r}} \quad (11.1)$$

In this expression  $r$  is the distance from the charge element  $dq$  to the point  $P$  where we are determining the electric field. The unit vector  $\hat{\mathbf{r}}$  points from  $dq$  to  $P$ .

By the superposition principle, the total electric field is the vector sum of all these infinitesimal contributions. (See Figure 11.1):



**Figure 11.1** Electric field due to  $dq$  at a point off the axis of the wire

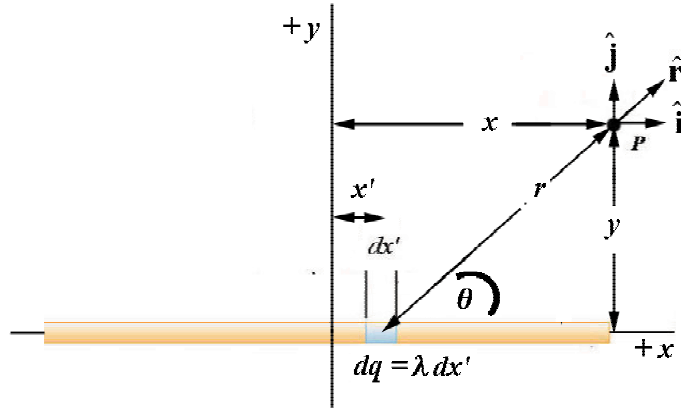
This sum is just the vector integral

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \int_{\text{wire}} \frac{dq}{r^2} \hat{\mathbf{r}} \quad (11.2)$$

Recall that a vector integral is actually three separate integrals, one for each direction in space that will give the component of the electric field in that direction. Each separate component integral is an integral over the length of the wire.

### Coordinate System:

In order to find explicit functions for the components of the electric field, we need to introduce a coordinate system. We choose Cartesian coordinates with the origin in the center of the wire. Define the  $x$ -axis to lie along the wire. We shall choose the  $y$ -axis so that the point  $P$  lies somewhere in the  $x$ - $y$  plane defined by  $z = 0$ . The coordinates for the point  $P$  are  $(x, y)$ . The unit vectors  $(\hat{\mathbf{i}}, \hat{\mathbf{j}})$  at the point  $P$  are shown in Figure 11.2.



**Figure 11.2** Coordinate system for wire

The charge element  $dq$  is located at a distance  $x'$  from the origin. The charge  $dq$  is the product of the charge density,  $\lambda$ , with the infinitesimal length  $dx'$ :

$$dq = \lambda dx' \quad (11.3)$$

The distance  $r$  from the charge element  $dq$  to the field point  $P$  is

$$r = ((x - x')^2 + y^2)^{1/2} \quad (11.4)$$

Finally the unit vector  $\hat{\mathbf{r}}$  from the source to the field point  $P$  can be decomposed as

$$\hat{\mathbf{r}} = \cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}} \quad (11.5)$$

where  $\theta$  is the angle that the unit vector  $\hat{\mathbf{r}}$  makes with the positive  $x$ -axis.

The contribution from  $dq$  to the electric field at the point  $P$  is given by

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{\lambda dx'}{(x - x')^2 + y^2} (\cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}}) \quad (11.6)$$

Then the total electric field at the point  $P$  is

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{\lambda dx'}{(x - x')^2 + y^2} (\cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}}) \quad (11.7)$$

We can now see explicitly the vector nature of our integral expression for the electric field. There are two separate integrals, one for the one for the  $x$ -direction and one for the  $y$ -direction,



$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{\lambda dx'}{(x-x')^2 + y^2} \cos\theta \hat{\mathbf{i}} + \frac{1}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{\lambda dx'}{(x-x')^2 + y^2} \sin\theta \hat{\mathbf{j}} \quad (11.8)$$

From Figure 11.4, we have the trigonometric relations

$$\cos\theta = \frac{x-x'}{((x-x')^2 + y^2)^{1/2}} \quad \sin\theta = \frac{y}{((x-x')^2 + y^2)^{1/2}} \quad (11.9)$$

The integrals then become

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{\lambda(x-x')dx'}{((x-x')^2 + y^2)^{3/2}} \hat{\mathbf{i}} + \frac{1}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{\lambda y dx'}{((x-x')^2 + y^2)^{3/2}} \hat{\mathbf{j}} \quad (11.10)$$

We need the integration formulas,

$$\int_{-L/2}^{L/2} \frac{\lambda(x-x')dx'}{((x-x')^2 + y^2)^{3/2}} = \frac{\lambda}{\sqrt{(x-x')^2 + y^2}} \Big|_{-L/2}^{L/2} = \frac{\lambda}{\sqrt{(x-L/2)^2 + y^2}} - \frac{\lambda}{\sqrt{(x+L/2)^2 + y^2}} \quad (11.11)$$

and

$$\int_{-L/2}^{L/2} \frac{\lambda y dx'}{((x-x')^2 + y^2)^{3/2}} = -\frac{1}{y} \frac{(x-x')}{\sqrt{(x-x')^2 + y^2}} \Big|_{-L/2}^{L/2} = \frac{1}{y} \left[ \frac{\lambda(x-L/2)}{\sqrt{(x-L/2)^2 + y^2}} - \frac{\lambda(x+L/2)}{\sqrt{(x+L/2)^2 + y^2}} \right] \quad (11.12)$$

Then the results of our integration is

$$\vec{\mathbf{E}}(x, y) = E_x(x, y) \hat{\mathbf{i}} + E_y(x, y) \hat{\mathbf{j}} \quad (11.13)$$

where the  $x$ -component of the electric field is

$$E_x(x, y) = \frac{\lambda}{4\pi\epsilon_0} \left[ \frac{1}{((x-L/2)^2 + y^2)^{1/2}} - \frac{1}{((x+L/2)^2 + y^2)^{1/2}} \right] \quad (11.14)$$

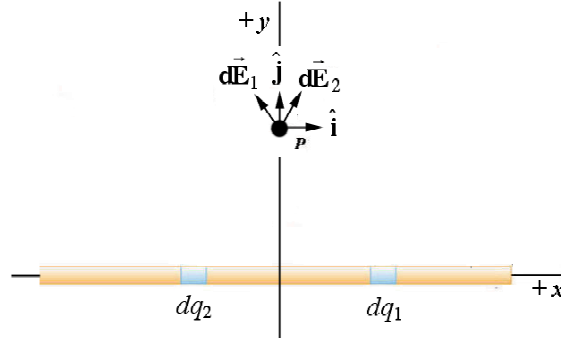
and the  $y$ -component of the electric field is

$$E_y(x, y) = \frac{\lambda}{4\pi\epsilon_0 y} \left[ \frac{x-L/2}{((x-L/2)^2 + y^2)^{1/2}} - \frac{x+L/2}{((x+L/2)^2 + y^2)^{1/2}} \right] \quad (11.15)$$

If we place our field point along the  $y$ -axis, with  $x = 0$ , the  $x$ -component of the electric field vanishes,

$$E_x(x, y) = \frac{1}{4\pi\epsilon_0} \left( \frac{\lambda}{((L/2)^2 + y^2)^{1/2}} - \frac{\lambda}{((L/2)^2 + y^2)^{1/2}} \right) = 0 \quad (11.16)$$

One way to see this is to add contributions from two charge elements that are mirror reflections of each other about the  $y$ -axis. Since they point in opposite directions, they add to zero (Figure 11.3).



**Figure 11.3** Vanishing of the  $x$ -component along the  $y$ -axis

The electric field is then

$$\vec{\mathbf{E}}(0, y) = \frac{1}{4\pi\epsilon_0} \frac{\lambda L}{\sqrt{(L/2)^2 + y^2}} \frac{1}{y} \hat{\mathbf{j}} \quad (11.17)$$

### Electric field due to an infinite uniformly charged wire:

We can also use our result to calculate the electric field due to an infinite wire with uniform charge density  $\lambda$  by taking the limit as the length,  $L$ , of the wire approaches infinity. We need the fact that

$$\lim_{L \rightarrow \infty} \frac{\lambda L}{y \sqrt{(L/2)^2 + y^2}} = \frac{2\lambda}{y} \quad (11.18)$$

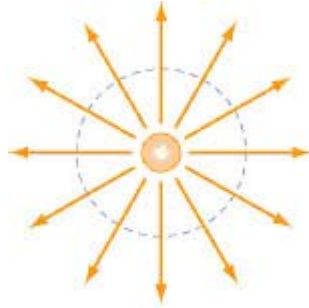
and

$$\lim_{L \rightarrow \infty} \frac{\lambda}{[(x - L/2)^2 + y^2]^{1/2}} - \frac{\lambda}{[(x + L/2)^2 + y^2]^{1/2}} = 0 \quad (11.19)$$

This shows that the  $x$ -component of the electric field vanishes for the infinite wire and the electric field of an infinite wire is

$$\vec{\mathbf{E}}(y) = \frac{\lambda}{2\pi\epsilon_0 y} \hat{\mathbf{j}} \quad (11.20)$$

This result shows us that the field falls off like  $y^{-1}$  and points either away or towards the wire depending on the whether the charge density  $\lambda$  is positive or negative.



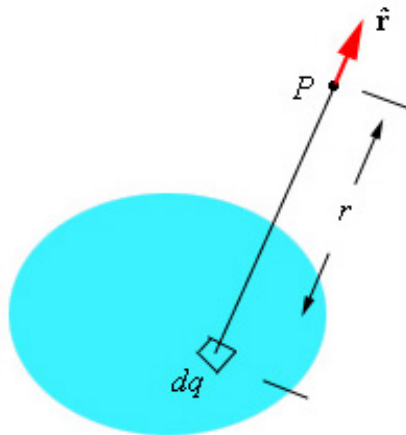
**Figure 11.4** Electric field of an infinite wire (perpendicular to the plane of the page)

### Example 12: Electric field due to a Uniformly Charged Disk

A disc of radius  $R$  is uniformly charged with total charge  $Q$ . The uniform surface charge density is  $\sigma = Q/(\pi R^2)$ . In order to find the electric field at a point  $P$ , along an axis that passes through the center of the disc perpendicular to the plane of the disc, we must consider a small charge element  $dq$  located on the disc. This element is the source for an infinitesimal electric field at the point  $P$  that is given by Coulomb's Law,

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \hat{r} \quad (12.1)$$

In this expression  $r$  is the distance from the charge element  $dq$  to the point  $P$  where we are determining the electric field. The unit vector  $\hat{r}$  points from the charge element  $dq$  to the point  $P$ . (See Figure 12.1).



**Figure 12.1** Charge element for the field of a disc

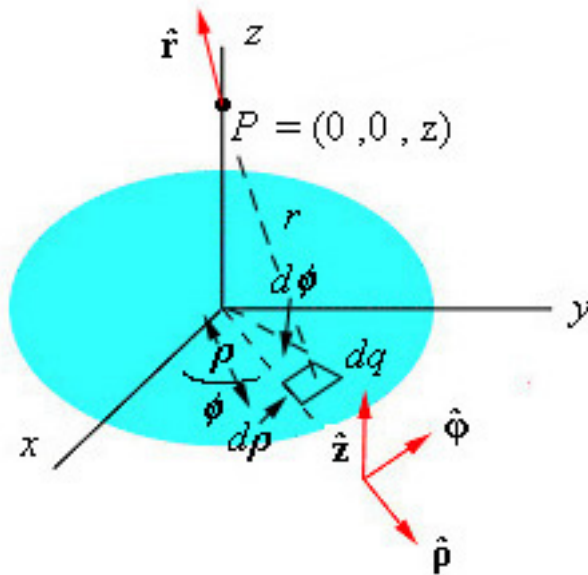
By the superposition principle, the total electric field is the vector sum of all these infinitesimal contributions. This sum is just the integral

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \int_{disc} \frac{dq}{r^2} \hat{\mathbf{r}} \quad (12.2)$$

This integral is an example of a vector integral over the surface of the disc. A vector integral is actually three separate integrals, one for each direction in space that will give the component of the electric field in that direction. Each separate component integral is an integral over the surface of a disk. It will turn out that only the non-zero component of the electric field will be the component in the direction parallel to the axis passing through the plane of the disk.

### Coordinate System:

To see this explicitly, we shall use polar coordinates. We choose polar coordinates because the geometry of disc has the same symmetry properties as polar coordinates. Define the  $z$ -axis to pass perpendicularly through the center of the disc. Let  $\hat{\mathbf{z}}$  be the unit vector pointing along the positive  $z$ -axis. Place the plane of the disc at  $z = 0$ . Choose an arbitrary point  $z$  on the axis for the field point  $P$  where we are trying to determine the electric field. Choose coordinates  $(\rho, \phi)$  for the plane of the disc. Let  $(\hat{\rho}, \hat{\phi})$  be the associated unit vectors. Note that  $\hat{\rho} \times \hat{\phi} = \hat{\mathbf{z}}$ .



**Figure 12.2** Coordinate system for a disc

The charge element  $dq$  is located at a distance  $\rho$  from the origin. The charge  $dq$  is the product of the charge density,  $\sigma$  and the area  $dA = \rho d\phi d\rho$ :

$$dq = \sigma dA = \sigma \rho d\phi d\rho \quad (12.3)$$

The distance  $r$  from the charge element  $dq$  to the field point  $P$  is

$$r = (z^2 + \rho^2)^{1/2} \quad (12.4)$$

Finally the unit vector  $\hat{\mathbf{r}}$  from the source to the field point  $P$  can be decomposed as

$$\hat{\mathbf{r}} = -\cos\theta \hat{\boldsymbol{\rho}} + \sin\theta \hat{\mathbf{z}} \quad (12.5)$$

where  $\theta$  is the angle that the unit vector  $\hat{\mathbf{r}}$  makes with the plane of the loop. The contribution from  $dq$  to the electric field at the point  $P$  is given by

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{\sigma \rho d\phi d\rho}{z^2 + \rho^2} (-\cos\theta \hat{\boldsymbol{\rho}} + \sin\theta \hat{\mathbf{z}}) \quad (12.6)$$

Then the total electric field at the point  $P$  is

$$\vec{\mathbf{E}} = \frac{\sigma}{4\pi\epsilon_0} \int_0^R \int_0^{2\pi} \frac{\rho d\phi d\rho}{z^2 + \rho^2} (-\cos\theta \hat{\boldsymbol{\rho}} + \sin\theta \hat{\mathbf{z}}). \quad (12.7)$$

We can now see explicitly the vector nature of our integral expression for the electric field. There are two separate integrals, one for the one for the  $\hat{\boldsymbol{\rho}}$  direction and one for the  $\hat{\mathbf{z}}$  direction,

$$\vec{\mathbf{E}} = -\frac{1}{4\pi\epsilon_0} \int_0^R \int_0^{2\pi} \frac{\sigma \rho d\phi}{z^2 + \rho^2} \cos\theta \hat{\boldsymbol{\rho}} + \frac{1}{4\pi\epsilon_0} \int_0^R \int_0^{2\pi} \frac{\sigma \rho d\phi d\rho}{z^2 + \rho^2} \sin\theta \hat{\mathbf{z}} \quad (12.8)$$

From Figure 12.2, we have the trigonometric relations

$$\cos\theta = \frac{\rho}{(z^2 + \rho^2)^{1/2}} \quad \sin\theta = \frac{z}{(z^2 + \rho^2)^{1/2}} \quad (12.9)$$

The integrals then become

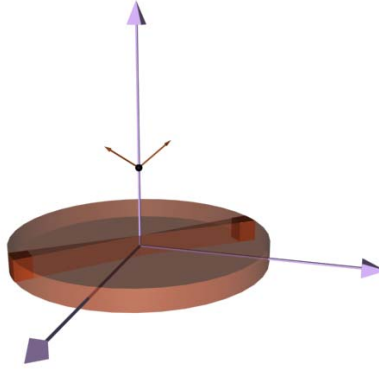
$$\vec{\mathbf{E}} = -\frac{1}{4\pi\epsilon_0} \int_0^R \int_0^{2\pi} \frac{\sigma \rho^2 d\phi d\rho}{(z^2 + \rho^2)^{3/2}} \hat{\boldsymbol{\rho}} + \frac{1}{4\pi\epsilon_0} \int_0^R \int_0^{2\pi} \frac{z \sigma \rho d\phi d\rho}{(z^2 + \rho^2)^{3/2}} \hat{\mathbf{z}} \quad (12.10)$$

The integration over  $d\phi$  is straightforward because the integrand is independent of  $\phi$ ,

$\int_0^{2\pi} d\phi = 2\pi$ . So our integral expression for the electric field becomes

$$\vec{\mathbf{E}} = -\frac{1}{2\epsilon_0} \int_0^R \frac{\sigma \rho^2 d\rho}{(z^2 + \rho^2)^{3/2}} \hat{\boldsymbol{\rho}} + \frac{1}{2\epsilon_0} \int_0^R \frac{\sigma z \rho d\rho}{(z^2 + \rho^2)^{3/2}} \hat{\mathbf{z}}. \quad (12.11)$$

The first integral vanishes. If we examine Figure 12.2, we see that the component of  $\vec{\mathbf{E}}$  in the  $\hat{\boldsymbol{\rho}}$  direction will add up to zero as we add up the contributions from each infinitesimal piece. The reason for this is that each little vector contribution points away from the center. Their sum adds to zero (Figure 12.3).



**Figure 12.3** Vector sum of the  $\hat{\boldsymbol{\rho}}$  direction

Another way to see this is to add contributions from two pieces directly opposite each other in the loop. Since they point in opposite directions, they add to zero. The third way to understand this is to decompose the unit vector  $\hat{\boldsymbol{\rho}}$  in terms of the Cartesian unit vectors  $\hat{\mathbf{i}}$  and  $\hat{\mathbf{j}}$ ,

$$\hat{\boldsymbol{\rho}} = \cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}} \quad (12.12)$$

Then the integral

$$\int_0^{2\pi} d\phi \hat{\boldsymbol{\rho}} = \int_0^{2\pi} d\phi \cos \phi \hat{\mathbf{i}} + \int_0^{2\pi} d\phi \sin \phi \hat{\mathbf{j}} \quad (12.13)$$

So the integral in the  $\hat{\boldsymbol{\rho}}$  direction is actually two integrals for the  $\hat{\mathbf{i}}$  and  $\hat{\mathbf{j}}$  directions. Each of these integrals separately vanishes. For example the  $x$ -component integral becomes

$$\int_0^{2\pi} d\phi \cos \phi \hat{\mathbf{i}} = \sin \phi \Big|_0^{2\pi} \hat{\mathbf{i}} = 0 \hat{\mathbf{i}} \quad (12.14)$$

A similar result holds for the  $y$ -component.

Thus only the  $z$ -component of the electric field is non-zero and our final result for the electric field along the axially axis is

$$\vec{\mathbf{E}} = \frac{1}{2\epsilon_0} \int_0^R \frac{\sigma z \rho d\rho}{(z^2 + \rho^2)^{3/2}} \hat{\mathbf{z}} \quad (12.15)$$

We need the integration formula,

$$\int_0^R \frac{\rho d\rho}{(z^2 + \rho^2)^{3/2}} = -\frac{1}{(z^2 + \rho^2)^{1/2}} \Bigg|_{\rho=0}^{\rho=R} = \frac{1}{z} - \frac{1}{(z^2 + R^2)^{1/2}} \quad (12.16)$$

Then the results of our integration is

$$\vec{\mathbf{E}} = \frac{\sigma}{2\epsilon_0} \left[ 1 - \frac{z}{(z^2 + R^2)^{1/2}} \right] \hat{\mathbf{z}}. \quad (12.17)$$

### Electric field very close to the surface of a disk:

We can also use our result to approximate the electric field when the field point is very close to the surface of the disk,  $z \ll R$ . We shall make a power series expansion using the binomial theorem for the term

$$\frac{z}{(z^2 + R^2)^{1/2}} = \frac{z}{R} \frac{1}{(1 + \frac{z^2}{R^2})^{1/2}} = \frac{z}{R} \left( 1 - \frac{1}{2} \frac{z^2}{R^2} + \frac{3}{8} \frac{z^4}{R^4} - \dots \right) \cong \frac{z}{R} \left( 1 - \frac{1}{2} \frac{z^2}{R^2} \right) \quad (12.18)$$

Then the electric field is near the surface of the plane is

$$\vec{\mathbf{E}} \cong \frac{\sigma}{2\epsilon_0} \left[ 1 - \frac{z}{R} \left( 1 - \frac{1}{2} \frac{z^2}{R^2} \right) \right] \hat{\mathbf{z}} \quad (12.19)$$

We can ignore terms proportional to  $z^3 / R^3$ , then

$$\vec{\mathbf{E}} \cong \frac{\sigma}{2\epsilon_0} \left[ 1 - \frac{z}{R} \left( 1 - \frac{1}{2} \frac{z^2}{R^2} \right) \right] \hat{\mathbf{z}} \cong \frac{\sigma}{2\epsilon_0} \left( 1 - \frac{z}{R} \right) \hat{\mathbf{z}} \quad (12.20)$$

Finally if we ignore terms proportional to  $z / R$ , we have

$$\vec{\mathbf{E}} \cong \frac{\sigma}{2\epsilon_0} \hat{\mathbf{z}} \quad (12.21)$$

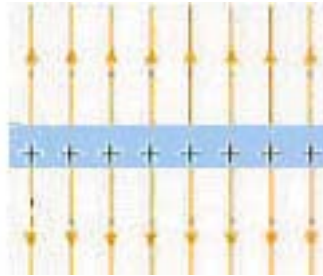
This shows that very near the disk, the magnitude of the electric field along the axial symmetry axis of the disk is approximately constant  $\sigma/(2\epsilon_0)$ .

### Electric field due to an infinite uniformly charged plane:

We can also use our result to calculate the electric field due to an infinite plane with uniform charge density  $\sigma$  by taking the limit as the radius,  $R$ , of the disk approaches infinity. Then the second term vanishes and the electric field along the  $z$ -axis of an infinite plane is

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{z} \quad (12.22)$$

This result shows us that the field is constant everywhere in space, (independent of the distance from the plane) (Figure 12.4).



**Figure 12.4** Electric field due to an infinite uniformly charged plane.

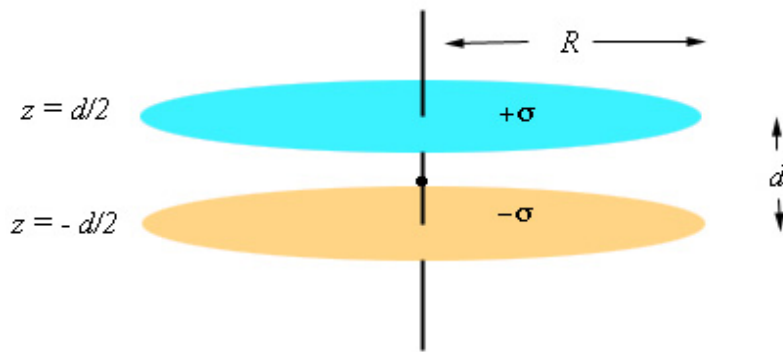
### Example 13: Electric field of two parallel discs:

Consider two parallel discs, each one of radius  $R$ , that are separated by a distance  $a$ . Each disc is uniformly charged with equal but opposite surface charge densities,  $\sigma_{\pm} = \pm\sigma$ , where  $\sigma = Q/(\pi R^2)$ . In order to find the total electric field everywhere in space we will add together the electric fields of the two oppositely charged discs (the superposition principle).

$$\vec{E}_{\text{total}} = \vec{E}_+ + \vec{E}_- \quad (13.1)$$

Let's assume that the separation between the discs is much smaller than the radius of the discs,  $d \ll R$ . Let's choose coordinates such that the plane of the discs is perpendicular to the  $z$ -axis, the positive disc is located at  $z = d/2$ , and the negative disc is located at  $z = -d/2$ .



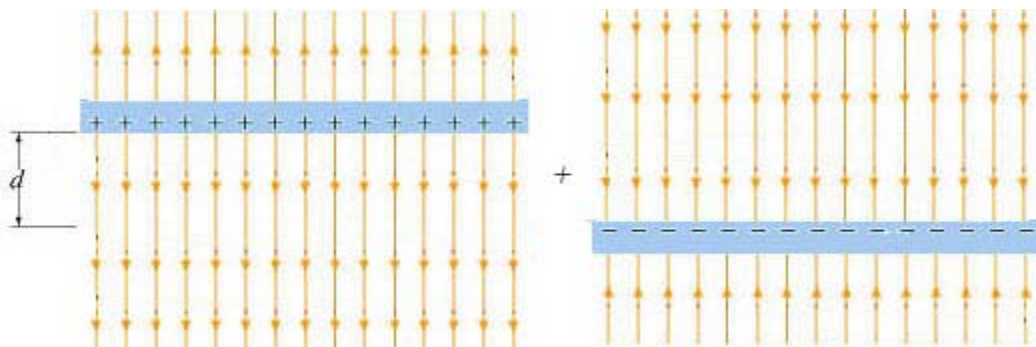


**Figure 13.1** Positive and negative discs

Then the electric field of the positive disc and negative discs are approximately uniform. The field of the positive disc points away from the positive disc and the field of the negative disc points towards the negative disc. The magnitudes of both fields are equal, because the discs carry equal but opposite surface charge densities,

$$E_+ = E_- \cong \frac{\sigma}{2\epsilon_0} \quad (13.2)$$

Therefore when we add these fields together, we see that the field outside the parallel discs is approximately zero, and the fields between the discs is twice the magnitude of the field of either disc.



**Figure 13.2** Approximate electric fields of oppositely charged parallel discs

The electric field of the positive disc is given by

$$\vec{\mathbf{E}}_+ = \begin{cases} +\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}, & z > d/2 \\ -\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}, & z < d/2 \end{cases} \quad (13.3)$$

The electric field of the negative disc is given by

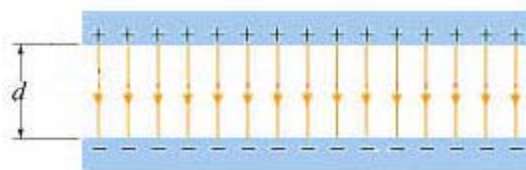
$$\vec{\mathbf{E}}_- = \begin{cases} -\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}, & z > -d/2 \\ +\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}, & z < -d/2 \end{cases} \quad (13.4)$$

When we add these two fields together we get

$$\vec{\mathbf{E}}_{\text{total}} = \begin{cases} 0\hat{\mathbf{z}}, & z > d/2 \\ -\frac{\sigma}{\epsilon_0}\hat{\mathbf{z}}, & d/2 > z > -d/2 \\ 0\hat{\mathbf{z}}, & z < -d/2 \end{cases} \quad (13.5)$$

So the electric field between the plates is twice the magnitude of the field of a single plate

$$\vec{\mathbf{E}}_{\text{total}} = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{z}}, \quad d/2 > z > -d/2.$$



**Figure 13.3** Approximate field between discs