**MASSACHUSETTS INSTITUTE OF TECHNOLOGY**
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**8.02** **Fall, 2002**

# V Capacitor - Worked Examples

### Example 1: Cylindrical Capacitor

Consider a solid cylindrical conductor of radius *a* surrounded by a coaxial cylindrical shell of inner radius *b*, as shown in Figure 1.1. The length of both cylinders is *l* and we take it to be much larger compared to *b-a*, the separation of the cylinders, so that edge effects can be neglected. The capacitor is charged so that the inner cylinder has charge $+Q$ while the outer shell has a charge $-Q$.

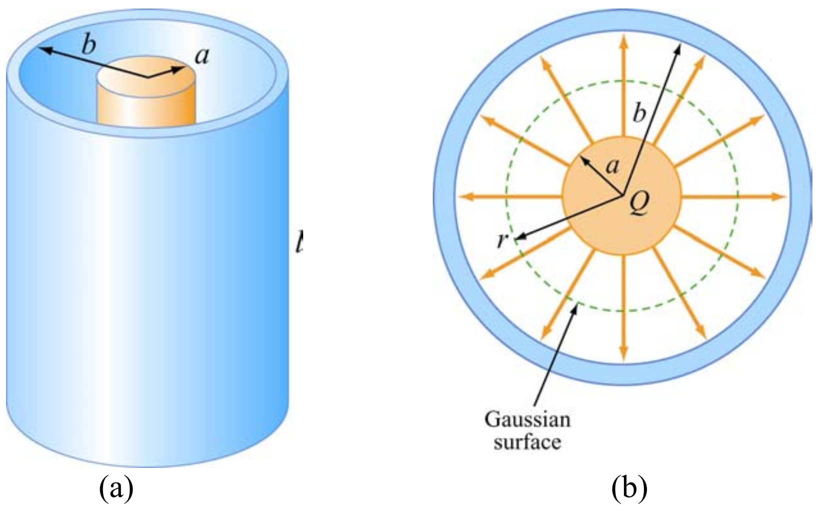



**Figure 1.1** (a) A cylindrical capacitor. (b) End view of the capacitor. The electric field is non-vanishing only in the region $a < r < b$.

To obtain the capacitance, we first compute the electric field. Using Gauss's law, we have

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E(2\pi r l) = \frac{Q}{\varepsilon_0} \qquad \Rightarrow \qquad E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

where $\lambda = Q/l$ is the charge/unit length. The potential difference can then be obtained as:

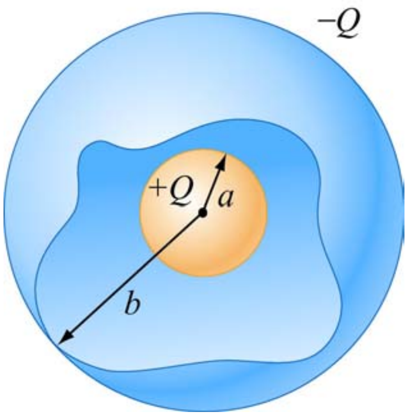
$$V_b - V_a = -\int_a^b dr E_r = -\frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{dr}{r} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$$
$$\Delta V = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right) = \frac{Q}{C} = \frac{\lambda l}{C} \tag{1.1}$$

which yields

$$\boxed{C = \frac{2\pi\varepsilon_0 l}{\ln\left(\frac{b}{a}\right)}} \tag{1.2}$$

**Example 2: Spherical Capacitor**

A spherical capacitor consists of two concentric spherical shells of radii *a* and *b*, as shown in Figure 2.1a. Figure 2.1b shows how the charging battery is connected to the capacitor. The inner shell has a charge $+Q$ uniformly distributed over its surface, and the outer shell an equal but opposite charge $-Q$.



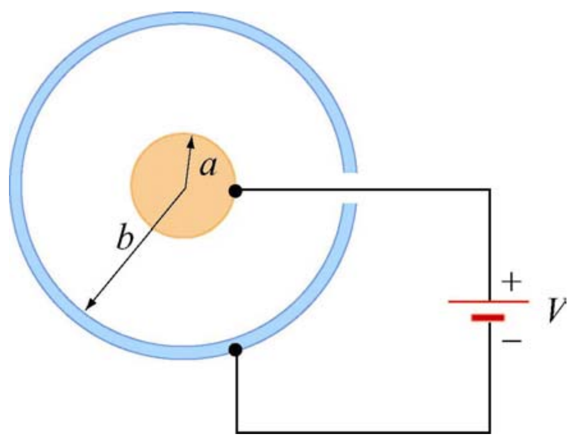




**Figure 2.1** (a) A spherical capacitor consisting of two concentric spherical shells of radii *a* and *b*. (b) Charging of the spherical capacitor

The capacitance of this configuration can be computed as follows: The electric field in the region $a < r < b$ is given by

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E_r A = E_r\left(4\pi r^2\right) = \frac{Q}{\varepsilon_0} \tag{2.1}$$

or

$$E_r = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2} \tag{2.2}$$

The potential difference between the two conducting shells is:

$$V_b - V_a = -\int_a^b dr E_r = -\frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right) \tag{2.3}$$

With $\Delta V = V_a - V_b$, we have

$$\boxed{C = \frac{Q}{\Delta V} = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)} \tag{2.4}$$

An isolated conductor can also have a capacitance. In the limit where $b \to \infty$, the above equation becomes

$$\lim_{b\to\infty} C = \lim_{b\to\infty} 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right) = \lim_{b\to\infty} 4\pi\varepsilon_0 \frac{a}{\left(1-\frac{a}{b}\right)} = 4\pi\varepsilon_0 a \tag{2.5}$$

Thus, for a single isolated spherical conductor of radius $R$,

$$\boxed{C = 4\pi\varepsilon_0 R.} \tag{2.6}$$

The above expression can also be obtained by noting that a sphere of radius $R$ has $V = \frac{Q}{4\pi\varepsilon_0}$, and $V=0$ at infinity. This yields

$$C = \frac{Q}{\Delta V} = \frac{Q}{Q/4\pi\varepsilon_0 R} = 4\pi\varepsilon_0 R. \tag{2.7}$$

As expected, the capacitance of an isolated charged sphere only depends on its geometry (the radius $R$).

**Example 3: Capacitor voltage divider**

The charge $Q$ on a capacitor $C$ is related to the voltage $V$ across it

$$Q = CV \tag{3.1}$$

Consider two capacitors, $C_1$ and $C_2$, in series across an alternating voltage source, $V = V_0 \sin(2\pi ft)$, as shown in Figure 3.1.

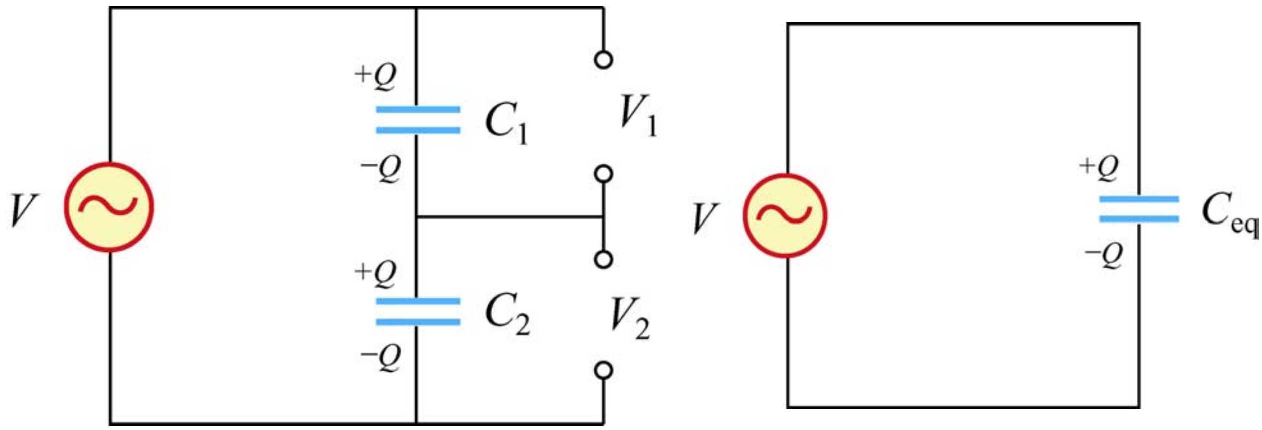



**Figure 3.1** Capacitor voltage divider

What is the voltage across $C_2$? The two capacitors in series look like a single capacitor $C_{eq} = \frac{C_1 C_2}{(C_1 + C_2)}$ as far as the voltage source is concerned. The same current, $I = \frac{dQ}{dt}$, flows through both capacitors and produces the same alternating charge on them.

The current is then

$$I = \frac{dQ}{dt} = C_{eq} \frac{dV}{dt} \tag{3.2}$$

So the alternating charge on the two capacitors becomes via integration

$$Q = \int I\, dt = C_{eq} V = C_1 V_1 = C_2 V_2 \tag{3.3}$$

where $V$ is the voltage across both capacitors in series, and $V_1$ and $V_2$ are the voltages across $C_1$ and $C_2$, respectively.

Solving for $V_2$ we get

$$V_2 = \frac{Q}{C_2} = C_{eq} \frac{V}{C_2} = \frac{V}{C_2} \left( \frac{C_1 C_2}{C_1 + C_2} \right) = V \left( \frac{C_1}{C_1 + C_2} \right) \tag{3.4}$$

The ratio $V_2/V$ describes the voltage divider and is given by

$$\frac{V_2}{V} = \frac{C}{C_1 + C_2} \tag{3.5}$$

We have what's called a capacitative voltage divider for ac voltage that works independently of frequency, at least in its ideal form.

In the HVPS (high voltage power supply), $C_1 = 100\,\text{pF}$ and $C_2 = 1000\,\text{pF}$, so the smaller voltage $V_2 = \frac{1}{11}V$ appears across the larger capacitor $C_2$ and the larger voltage $V_1 = \frac{10}{11}V$ appears across the smaller capacitor $C_1$—just the opposite of a resistive voltage divider (or pot) which you have seen and used before, where the larger voltage appears across the larger resistor.