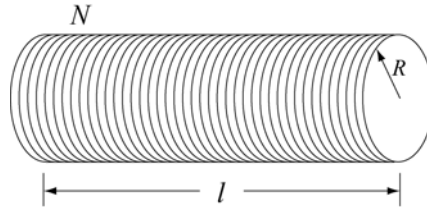


## XI. Inductance - Worked Examples

### Example 1: Solenoid

A long solenoid with length  $l$  and a radius  $R$  consists of  $N$  turns of wire, as shown in the figure below.



- (a) Neglecting the end effects, find the self-inductance.  
(b) A current  $I$  is passed through the coil. Find the energy stored in the system.

### Solution:

- (a) Using Ampere's law, the magnetic field inside a solenoid is

$$B = \frac{\mu_0 N I}{l} = \mu_0 n I \quad (1.1)$$

where  $n$  is the number of turns per unit length. The magnetic flux through each turn is

$$\Phi_B = BA = \mu_0 n I \cdot (\pi R^2) = \mu_0 n I \pi R^2 \quad (1.2)$$

Thus, the self-inductance is

$$L = \frac{N \Phi_B}{I} = \mu_0 n^2 \pi R^2 l \quad (1.3)$$

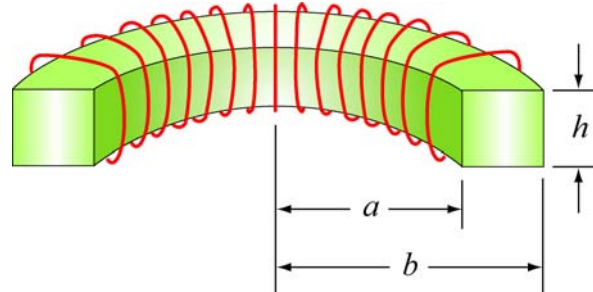
- (b) The energy of the system is given by

$$U_B = \frac{1}{2} L I^2 = \frac{1}{2} \mu_0 n^2 I^2 \pi R^2 l = \frac{1}{2 \mu_0} (\mu_0 n I)^2 (\pi R^2 l) = \frac{B^2}{2 \mu_0} (\pi R^2 l) \quad (1.4)$$

Since  $\pi R^2 l$  corresponds to the volume of the solenoid,  $u_B = B^2 / 2 \mu_0$  represents the magnetic energy density.

### Example 2: Toroid

A toroid consists of  $N$  turns and has a rectangular cross section, with inner radius  $a$ , outer radius  $b$  and height  $h$  (see figure).

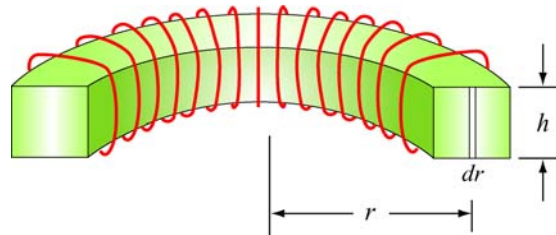


(a) Find its self-inductance  $L$ .

(b) Find the total magnetic energy stored in the toroid.

#### Solution:

(a) To find the self-inductance, we first need to know the magnetic field everywhere. From symmetry consideration, the magnetic field inside the toroid must be circular, i.e., along the azimuthal direction. Therefore, in applying Ampere's law, we choose the integration path to be a circle of radius  $r$ .



This gives

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\pi r) = \mu_0 I_{enc} = \mu_0 NI \quad (2.1)$$

or

$$B = \frac{\mu_0 NI}{2\pi r} \quad (2.2)$$

The magnetic flux through one turn of the toroid may be obtained by integrating over the rectangular cross section, with  $dA = h dr$  as the differential area element:

$$\Phi_B = \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = \int_a^b \left( \frac{\mu_0 NI}{2\pi r} \right) h dr = \frac{\mu_0 NI h}{2\pi} \ln \left( \frac{b}{a} \right) \quad (2.3)$$

The total flux is  $N\Phi_B$ . Therefore, the self-inductance is

$$L = \frac{N\Phi_B}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right) \quad (2.4)$$

(b) The energy density of the magnetic field is

$$u_B = \frac{1}{2} \frac{B^2}{\mu_0} = \frac{\mu_0 N^2 I^2}{8\pi^2 r^2} \quad (2.5)$$

The total energy stored in the magnetic field can be found by integrating over the volume. We choose the differential volume element to be a ring with radius  $r$ , width  $dr$  and height  $h$ , i.e.,  $dV = 2\pi r h dr$ . This yields

$$U_B = \int u_B dV = \int_a^b \left( \frac{\mu_0 N^2 I^2}{8\pi^2 r^2} \right) 2\pi r h dr = \frac{\mu_0 N^2 I^2 h}{4\pi} \ln \frac{b}{a} \quad (2.6)$$

Since  $U_B = \frac{1}{2} LI^2$ , the self-inductance may also be obtained as

$$L = \frac{2U_B}{I^2} = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a} \quad (2.7)$$

which agrees with the result obtained in part (a).

### Example 3: Magnetic energy density

A wire of nonmagnetic material with radius  $R$  and length  $l$  carries a current  $I$  which is uniformly distributed over its cross-section. What is the magnetic energy per unit length inside the wire?

#### Solution:

Applying the Ampere's law, the magnetic field at distance  $r \leq R$  can be obtained as:

$$B(2\pi r) = \mu_0 J(\pi r^2) = \mu_0 \left( \frac{I}{\pi R^2} \right) (\pi r^2) \quad (3.1)$$

which yields

$$B = \frac{\mu_0 I r}{2\pi R^2} \quad (3.2)$$

Since the magnetic energy density (energy per unit volume) is given by

$$u_B = \frac{B^2}{2\mu_0} \quad (3.3)$$

the total energy stored in the system becomes

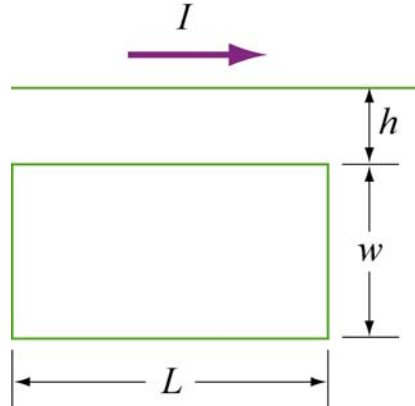
$$U = \int_0^R \frac{B^2}{2\mu_0} (2\pi r l dr) = \frac{\mu_0 I^2 l}{4\pi R^4} \int_0^R r^3 dr = \frac{\mu_0 I^2 l}{4\pi R^4} \left( \frac{R^4}{4} \right) \quad (3.4)$$

Thus, the magnetic energy per unit length is equal to

$$U_B / l = \frac{\mu_0 I^2}{16\pi} \quad (3.5)$$

#### Example 4: Mutual inductance

An infinite straight wire carrying current  $I$  is placed above a rectangular loop of wire with width  $w$  and length  $L$ , as shown in the figure below.



Determine the mutual inductance of the system.

#### Solution:

To calculate the mutual inductance  $M$ , we first need to know the magnetic flux through the rectangular loop. The magnetic field at a distance  $r$  away from the straight wire is  $B = \mu_0 I / 2\pi r$ , using Ampere's law. The total magnetic flux  $\Phi_B$  through the loop can be obtained by summing over contributions from all differential area elements  $dA = L dr$ :

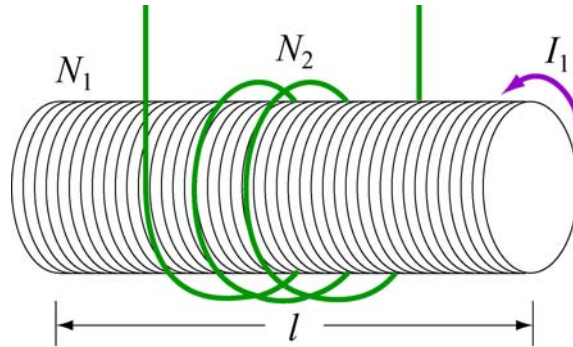
$$\Phi_B = \int d\Phi_B = \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = \frac{\mu_0 I L}{2\pi} \int_h^{h+w} \frac{dr}{r} = \frac{\mu_0 I L}{2\pi} \ln\left(\frac{h+w}{h}\right) \quad (4.1)$$

Thus, the mutual inductance is

$$\boxed{M = \frac{\Phi_B}{I} = \frac{\mu_0 L}{2\pi} \ln\left(\frac{h+w}{h}\right)} \quad (4.2)$$

### Example 5: Mutual inductance

A long solenoid with length  $l$  and a radius  $R$  consists of  $N_1$  turns of wire. If an insulated coil of  $N_2$  turns is wrapped around it, calculate the mutual inductance, assuming that all the flux from the solenoid passes through coil 2.



### Solution:

The magnetic flux through each turn of the second coil due to the solenoid is

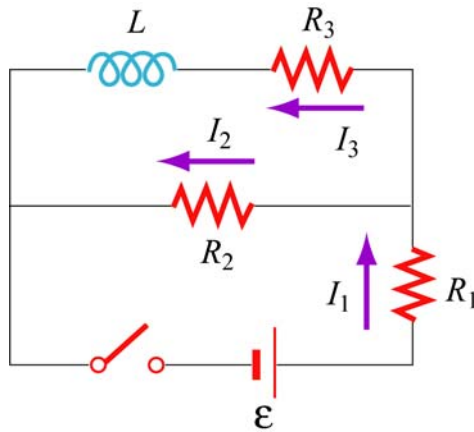
$$\Phi_{21} = BA = \frac{\mu_0 N I_1}{l} A \quad (5.1)$$

where  $B = \mu_0 N I_1 / l$  is the uniform magnetic field inside the solenoid. Thus, the mutual inductance is

$$M = \frac{N_2 \Phi_{21}}{I_1} = \frac{\mu_0 N_1 N_2 A}{l} \quad (5.2)$$

### Example 6: RL circuit

Consider the circuit shown in the figure below.



Determine the current through each resistor

(a) immediately after the switch is closed

(b) a long time after the switch is closed.

Suppose the switch is reopened a long time after it's been closed, what is each current

(c) right after it is opened?

(d) after a long time?

**Solution:**

(a) Immediately after the switch is closed, the current through the inductor is zero due to the induced emf.

Therefore,  $I_3 = 0$ . Since  $I_1 = I_2 + I_3$ , we have  $I_1 = I_2$

Applying Kirchhoff's rules to the first loop yields

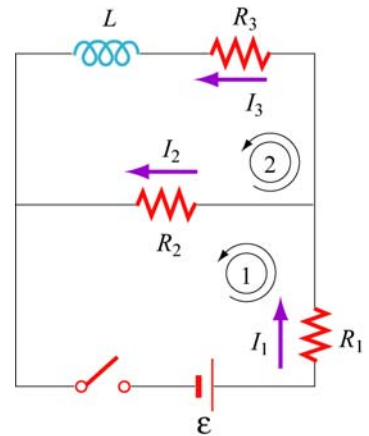
$$I_1 = I_2 = \frac{\varepsilon}{R_1 + R_2} \quad (6.1)$$

(b) After the switch has been closed for a long time, there is no induced emf in the inductor and the currents will be constant. Kirchhoff's loop (voltage) rule gives

$$\varepsilon - I_1 R_1 - I_2 R_2 = 0 \quad (6.2)$$

for the first loop, and

$$I_2 R_2 - I_3 R_3 = 0 \quad (6.3)$$



for the second. Combining the two equations with the junction rule  $I_1 = I_2 + I_3$ , we obtain

$$\begin{aligned} I_1 &= \frac{(R_2 + R_3)\varepsilon}{R_1R_2 + R_1R_3 + R_2R_3} \\ I_2 &= \frac{R_3\varepsilon}{R_1R_2 + R_1R_3 + R_2R_3} \\ I_3 &= \frac{R_2\varepsilon}{R_1R_2 + R_1R_3 + R_2R_3} \end{aligned} \tag{6.4}$$

(c) Immediately after the switch is opened, the current through  $R_1$  is zero, i.e.,  $I_1 = 0$ . This implies that  $I_2 + I_3 = 0$ . On the other hand, loop 2 now forms a decaying  $RL$  circuit and  $I_3$  starts to decrease. Thus,

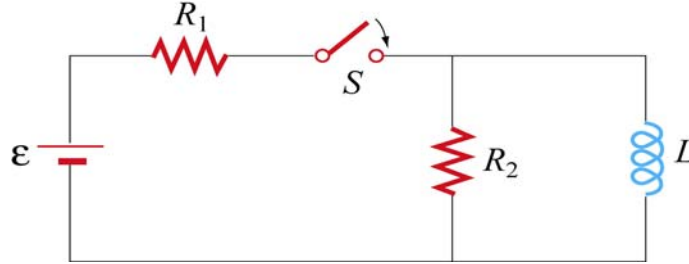
$$I_3 = -I_2 = \frac{R_2\varepsilon}{R_1R_2 + R_1R_3 + R_2R_3} \tag{6.5}$$

(d) A long time after the switch has been closed, all currents will be zero, i.e.,  $I_1 = I_2 = I_3 = 0$ .



### Example 7: RL circuit

In the circuit shown below, suppose the circuit is initially open. At time  $t = 0$  it is thrown closed. What is the current in the inductor at a later time  $t$ .



#### Solution:

Let the currents through  $R_1$ ,  $R_2$  and  $L$  be  $I_1$ ,  $I_2$  and  $I$ , respectively. From the Kirchhoff's junction rule, we have  $I_1 = I_2 + I$ . Applying the Kirchhoff's voltage rule to the left loop yields

$$\varepsilon - (I + I_2)R_1 - I_2R_2 = 0 \quad (7.1)$$

Similarly, for the outer loop, we have

$$\varepsilon - (I + I_2)R_1 = L \frac{dI}{dt} \quad (7.2)$$

The two equations can be combined to give

$$I_2R_2 = L \frac{dI}{dt} \quad (7.3)$$

which implies

$$I_2 = \frac{L}{R_2} \frac{dI}{dt} \quad (7.4)$$

Substituting into the first equation the above expression for  $I_2$ , we obtain

$$\varepsilon - \left( I + \frac{L}{R_2} \frac{dI}{dt} \right) R_1 - L \frac{dI}{dt} = 0 \quad (7.5)$$

which can be simplified to be

$$\varepsilon - IR_1 - \left( \frac{R_1 + R_2}{R_2} \right) L \frac{dI}{dt} = 0 \quad (7.6)$$

Dividing the equation by  $(R_1 + R_2)/R_2$  leads to

$$\varepsilon' - IR' - L \frac{dI}{dt} = 0 \quad (7.7)$$

where

$$R' = \frac{R_1 R_2}{R_1 + R_2}, \quad \varepsilon' = \frac{R_2 \varepsilon}{R_1 + R_2} \quad (7.8)$$

The differential equation can be solved and the solution is given by

$$I(t) = \frac{\varepsilon'}{R'} (1 - e^{-R't/L}) \quad (7.9)$$

Since

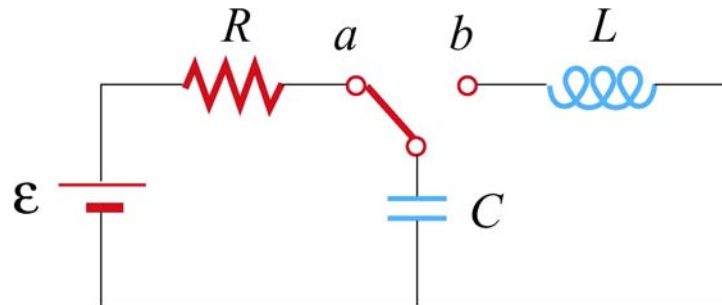
$$\frac{\varepsilon'}{R'} = \frac{\varepsilon R_2 / (R_1 + R_2)}{R_1 R_2 / (R_1 + R_2)} = \frac{\varepsilon}{R_1} \quad (7.10)$$

the current through the inductor may be rewritten as

$$I(t) = \frac{\varepsilon}{R_1} (1 - e^{-R't/L}) \quad (7.11)$$

### Example 8: LC circuit

Consider the circuit below. Suppose the switch which has been connected to point  $a$  for a long time is suddenly thrown to  $b$  at  $t = 0$ .



Find the following:

- the frequency of oscillation of the  $LC$  circuit.
- the maximum charge that appears on the capacitor.
- the maximum current in the inductor.
- total energy the circuit possesses at any time  $t$ ?

#### Solution:

- (a) The (angular) frequency of oscillation of the  $LC$  circuit is given by  $\omega = 2\pi f = 1/\sqrt{LC}$ . Therefore, the frequency is

$$f = \frac{1}{2\pi\sqrt{LC}} \quad (8.1)$$

- (b) The maximum charge stored in the capacitor before the switch is thrown to  $b$  is

$$Q = C\mathcal{E} \quad (8.2)$$

- (c) The energy stored in the capacitor before the switch is thrown is

$$U_c = \frac{1}{2}C\mathcal{E}^2 \quad (8.3)$$

On the other hand, the magnetic energy stored in the inductor is

$$U_B = \frac{1}{2}LI^2 \quad (8.4)$$

Thus, when the current is at its maximum, all the energy originally stored in the capacitor is now in the inductor:

$$\frac{1}{2}C\varepsilon^2 = \frac{1}{2}LI_{\max}^2 \quad (8.5)$$

This implies

$$I_{\max} = \varepsilon\sqrt{\frac{C}{L}} \quad (8.6)$$

(d) At any time, the total energy in the circuit would be equal to the initial energy that the capacitance stored, that is

$$U = U_C + U_B = \frac{1}{2}C\varepsilon^2 \quad (8.7)$$