8.022 Lecture supplement on resistors and circuits

McGreevy, 03.05.08

1 Resistivity and resistance

Consider a spherical shell made of a material of conductivity σ with inner radius a, outer radius b. (Conductivity σ means resistivity $1/\sigma$.) We can assume that the material satisfies ohm's law, namely that when some electric field $\vec{E}$ is applied to it, a current $\vec{J} = \sigma\vec{E}$ flows through it. We are asked to find the *resistance* of this object to current flowing from inside it to outside it.

Let's begin by determining what happens when there is some potential difference between the inside and outside, say the potential has spherical symmetry and $\phi(a) = V, \phi(b) = 0$.

The potential which satisfies Laplace's equation inside the shell[1] is of the form

$$\phi(r) = A + B/r.$$

A and B are determined by the boundary conditions to give

$$\phi(r) = V\left(\frac{ab}{b-a}\frac{1}{r} - \frac{a}{b-a}\right).$$

Taking the gradient

$$\vec{E} = -\vec{\nabla}\phi = \hat{r}V\frac{ab}{b-a}\frac{1}{r^2}$$

and using

$$\vec{J} = \sigma\vec{E} = \sigma V\hat{r}\frac{ab}{b-a}\frac{1}{r^2},$$

finally the current through the object is

$$I = \int \vec{J}\cdot d\vec{a} = \vec{J}\cdot\vec{A}$$

[1]Why should it satisfy the Laplace equation inside? Well,

$$4\pi\rho = \vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(\vec{J}/\sigma)$$

and σ is constant so this is

$$\rho \propto \vec{\nabla}\cdot J = \frac{\partial\rho}{\partial t}$$

but ρ is time-independent if we have a steady current, so this is zero. All the charge goes to the boundary.

where in the last step we used the fact that the current has spherical symmetry and so the integral is trivial. This is

$$I = |J| 4\pi \frac{b^2}{b^2} \frac{ab}{b-a} \sigma V.$$

Comparing to the macroscopic Ohm's law, we see

$$R = V/I = \frac{b-a}{4\pi\sigma ab}.$$

2 Power dissipated in resistors

The presence of flowing current means charges are moving through a potential difference V, which means work is being done

$$dw = V dq.$$

Power is defined to be work done per unit time, which says

$$P = \frac{dw}{dt} = V\frac{dq}{dt} = VI.$$

The power dissipated by a resistor is then

$$P = VI = I^2 R.$$

3 Warning: Not all materials are ohmic

Ohm's law is not the same kind of law as Gauss' Law – it's not as true. Particularly when electric fields are very large, the resulting current need not be linear in E. The existence of sparks provides a good example.

4 Capacitors introduce time-dependence

In a problem with current flowing through a circuit with a capacitor, both Q and $I = \frac{dQ}{dt}$ appear, so $I \neq 0$ means $Q = Q(t)$.

Suppose we have charge $\pm Q_0$ on the plates of a capacitor with capacitance C. At time $t = 0$ we close a switch which completes a circuit including the capacitor and a resistor with resistance R. What happens?

In words, the difference in potential created by the separation of charge across the capacitor pushes charges through the resistor, and decreases the charge on the capacitor.

More quantitatively, we reason as follows. The initial charge on the capacitor is $V(0) = Q_0/C$. At $t = 0$, the current will satisfy

$$I(0) = V(0)/R = Q/RC.$$

But $I = -\frac{dQ}{dt}$; the minus sign arises because the charge on the capacitor is decreasing as a result of the current flow. Therefore we have the nice differential equation

$$\frac{dQ}{dt} = -\frac{Q}{RC}.$$

To solve this, put all the stuff that depends on Q on one side:

$$\frac{dQ}{Q} = -\frac{dt}{RC}.$$

Notice that RC has units of time. Now we can integrate the BHS (both hand side) of this equation and get

$$\int_{Q_0}^{Q} \frac{dQ'}{Q'} = \int_0^t \frac{-dt}{RC}$$

The LHS is

$$\ln Q - \ln Q_0 = \ln Q/Q_0$$

The RHS is

$$-\frac{t}{RC}$$

Exponentiating the BHS we get

$$Q(t) = Q_0 e^{-t/RC}.$$

So the time RC is how long it takes the charge on the capacitor to decrease by a factor of $1/e$.

What's the current during all this drama?

$$I = -\frac{dQ}{dt} = +\frac{Q_0}{RC} e^{-t/RC}.$$

5 Star-triangle duality

This problem is based on Purcell 4.20. Given a black box with three electrodes labelled a, b, c sticking out of it, you measure the resistances R_{ab}, R_{bc}, R_{ac} between them by putting a potential difference across them in pairs and measuring the current.

a) What's inside?

In configuration one, three resistors with resistances $R_{1,2,3}$ meet at a vertex. Then

$$R_{ab} = R_1 + R_2$$

$$R_{bc} = R_2 + R_3$$

$$R_{ac} = R_1 + R_3$$

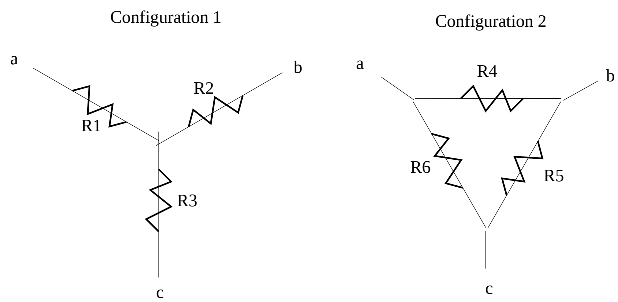

Figure 1: Two possibilities for the contents of the box.

In configuration two, the three resistors $R_{4,5,6}$ make a triangle, and

$$R_{ab} = \frac{(R_5 + R_6)R_4}{R_4 + R_5 + R_6}$$

$$R_{bc} = \frac{(R_4 + R_6)R_5}{R_4 + R_5 + R_6}$$

$$R_{ac} = \frac{(R_4 + R_5)R_6}{R_4 + R_5 + R_6}$$

By equating the pairs of expressions we can make the two systems equivalent. To solve for R_1 in terms of $R_{3,4,5}$, for example, add $R_{ab} + R_{bc} - R_{ac} = 2R_1$ from the first system, and from the second system,

$$R_{ab} + R_{bc} - R_{ac} = \frac{(R_5 + R_6)R_4 + (R_4 + R_6)R_5 + (R_4 + R_5)R_6}{R_4 + R_5 + R_6} = \frac{2R_4R_6}{R_4 + R_5 + R_6}$$

So $R_1 = \frac{R_4R_6}{R_4+R_5+R_6}$.

b) If you also measure the potential at the third node when putting a potential across the other two can you disambiguate?

No, you still can't. In configuration 1, the voltage drop from a to c is simply $V_c - V_a = -I_1R_1 - I_3R_3$ but no current flows through R_3, so this is

$$V_{ca}^{(1)} = -I_1R_1 = -IR_1 = -\frac{V_{ab}}{R_1 + R_2}R_1.$$

In configuration two, the potential at c is

$$V_c = V_a - I_6R_6$$

where I_6 is the current through R_6. But this satisfies

$$V_{ba} = -I_6(R_5 + R_6)$$

since this provides a path from a to b. So $I_6 = \frac{V_{ba}}{R_5+R_6}$ and we have

$$V_{ca}^{(2)} = -I_6R_6 = -\frac{V_{ba}}{R_5 + R_6}R_6.$$

Using our solution for R_1 above, we have

$$\frac{V_{ca}^{(1)}}{V_{ba}} = \frac{R_1}{R_1 + R_2} = \frac{R_4 R_6}{R_4 + R_5 + R_6} \frac{R_4 + R_5 + R_6}{(R_5 + R_6) R_4} = \frac{R_6}{R_5 + R_6} = \frac{V_{ca}^{(2)}}{V_{ba}}$$

So even with this extra data, you can't tell the difference, you just have to look inside.

6 Capacitors in parallel

Given two capacitors with capacitances $C_{1,2}$ in parallel, how can we pretend that this is a single capacitor?

Argument one: The charge Q on a capacitor with capacitance C with a potential difference V across it is $Q = CV$ (definition of C). For both capacitors C_1 and C_2, the potential difference is the same, call it V. So $Q_1 = C_1 V, Q_2 = C_2 V$. The equivalent capacitor must have the same total charge on it at a given potential difference, so $Q = Q_1 + Q_2 = V(C_1 + C_2) = VC_{eq}$ so $C_{eq} = C_1 + C_2$.

Argument two: Pretend they are made of parallel plates of area $A_{1,2}$ and some separation d . Then $C_1 = \xi A_1, C_2 = \xi A_2$ where ξ is something that depends on the dielectric constant of the stuff between the plates, and the distance between the plates and π and 2 and stuff. Then putting the capacitors in parallel is just the same as adding the areas, because they are made of conductors just like the wire connecting them. So $C_{eq} = \xi(A_1 + A_2)$.

Exercise for the reader: give the analogous arguments for capacitors in series.

7 Wheatstone bridge

A circuit like the one in Figure 2 is called a wheatstone bridge. It's notable because you can't reduce it just by using the formulas for resistors in parallel and in series. Call the current going into the wheatstone bridge I. What's I_3?

The first step in solving a resistor network is to add together resistors that are in series. So we get an equivalent slightly simpler thing. A steady current only changes at junctions. This fact has been used in labelling the figure.

Now use current conservation at every node:

$$I = I_4 + I_5 \tag{1}$$

$$I = I_1 + I_2 \tag{2}$$

$$I_1 = I_3 + I_4 \tag{3}$$

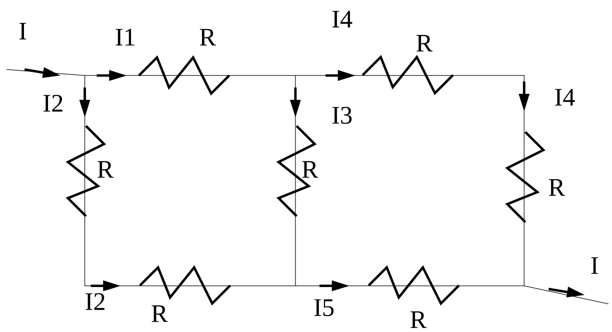

Figure 2: A wheatstone bridge circuit.

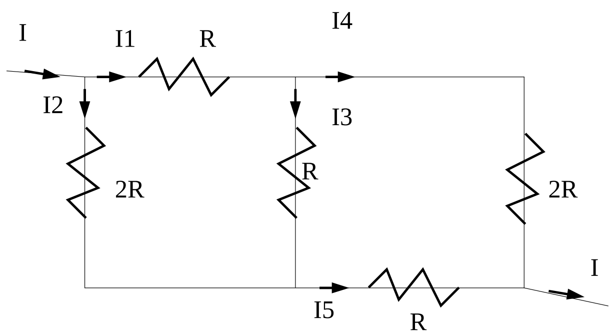

Figure 3: Simplified circuit.

The others give equations which are not independent of these.

Now use the fact that the potential is well-defined, or equivalently that the total voltage drop around a loop is zero. This is energy conservation (the circuit doesn't spontaneously generate current) or curl-free-ness of the electric field. In loop one,

$$0 = -2RI_2 + RI_3 + RI_1. \tag{4}$$

In loop two,

$$0 = I_3R + I_5R - I_4 2R. \tag{5}$$

Note that at each step above it is extremely easy to make a silly sign or labelling error which will totally mess up the final answer. Now we have an algebra problem, to solve five equations for five unknowns ($I_1 ... I_5$). It's horrible but doable given sufficient patience (or given Mathematica, where the command is **Solve[{ I4 + I5 == X, I1+I2==X, I3 + I4 ==I1, -2 I2 + I3 + I1 ==0, I3 + I5 - 2 I4 ==0}, {I1,I2,I3,I4,I5}]** – I've called the current I 'X' here because Mathematica uses 'I' for the square root of -1.).

Proceeding the old-fashioned way, eliminate $I_{1,3,5}$ using equations (1,2,3), to get

$$I_5 = I - I_4,\ I_1 = I - I_2,\ I_3 = I_1 - I_4 = I - I_2 - I_4$$

Plugging these into equation (4) simplifies to

$$I_4 = 2I - 4I_2.$$

Equation (5) then gives

$$I_2 = \frac{2}{5}I.$$

Finally, we end up with $I_3 = I/5$. Who would have guessed.

8 RC circuit

What happens if we stick a capacitor of capacitance C into the circuit on the leg labelled $I4$?

All the equations from Kirchoff's laws are the same except for Eqn. 5, which is now

$$0 = I_3R + I_5R - 2RI_4 - Q/C \quad (5')$$

I've labelled the charge Q on the capacitor so that $I_4 = -\frac{dQ}{dt}$. Eliminating everyone except I_4, I get

$$0 = -\frac{Q}{C} - \frac{17}{2}RI + \frac{15}{4}RI_4 = -\frac{Q}{C} - \frac{17}{2}RI - \frac{15}{4}R\frac{dQ}{dt}.$$

This is an equation of the form

$$\alpha\frac{dQ}{dt} + \beta Q = \gamma$$

where α, β, γ are constants. Its solution is a solution of the equation with $\gamma = 0$ plus some other solution.

$$Q(t) = \delta e^{-\frac{\beta}{\alpha}t} + \frac{\gamma}{\beta}$$

where δ some other constant. (Plug in and check.) The boundary condition $Q(t) = 0$ says that $\delta = -\frac{\gamma}{\beta}$.