

8.022 Lecture Supplement on Induction

McGreevy, 03.16.08

1 How much charge is moved by varying B ?

[Purcell 7.18] Suppose we have a loop of wire with area a and N turns attached through twisted leads to some resistor with total resistance R . It's sitting in a uniform magnetic field of strength B at time t_0 . Then we in some circuitous way turn off the B field (pull the plug on the electromagnet) so that at time t_1 the B field is zero. How much charge went through the resistor?

$$Q = \int_{t_0}^{t_1} I dt = \int \frac{\mathcal{E}(t)}{R} dt$$
$$\mathcal{E} = -\frac{1}{c} \int B \cdot da = -\frac{1}{c} N B a$$
$$Q = \frac{1}{R} \int_{t_0}^{t_1} \frac{\partial}{\partial t} \left(-\frac{N B a}{c} \right) = -\frac{a}{R c} N (B(t_1) - B(t_0)).$$

Note that this doesn't depend on the rate at which we turn off \vec{B} .

2 Terminal velocity, work done *via* magnetic field

[based on Purcell 7.5]

Consider the setup with a U-shaped wire with a movable contact rail which slides along the 'U'. Imagine hanging the 'U' upside down near the surface of the Earth, in the presence of some horizontal $\vec{B} = B\hat{x}$ magnetic field. Say the width of the 'U' is L , and the total resistance in the loop, dominated by the resistance in the rail, is R .

a) At what velocity will the rail end up moving?

If the rail is falling at a speed $v = -\dot{z}$, Faraday tells us that the EMF in the loop is

$$\mathcal{E} = -\frac{1}{c} \frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{a} = -\frac{1}{c} \frac{\partial}{\partial t} (B L z(t)) = -\frac{B L v}{c}.$$

The current is

$$I = \frac{\mathcal{E}}{R} = \frac{BLv}{cR}$$

(in a direction so that the B field it creates opposes the background B field; note that we are ignoring this B field here.). The magnetic force on the rail with current I is

$$F_{mag} = \int \frac{I}{c} \vec{B} \times d\vec{l} = \frac{xIB}{c}$$

(upwards). Plugging in I ,

$$F_{mag} = \frac{B^2 L^2 v}{c^2 R}$$

(note that I lost a factor of L in a similar place in lecture on Monday, March 17, 2008). The terminal velocity v_c occurs when this equals the downwards force of gravity:

$$mg = F_{grav} = F_{mag} = \frac{B^2 L^2 v}{c^2 R}$$

giving

$$v_c = \frac{mgc^2 R}{B^2 L^2}.$$

If the velocity is bigger than this, the magnetic force will slow the rail down; if the velocity is bigger, gravity will accelerate it. Either way, it ends up at the terminal value. More precisely, the acceleration of the rail satisfies

$$ma = m \frac{\partial v}{\partial t} = F_{grav} - F_{mag} = mg - m\alpha v$$

where $\alpha \equiv \frac{B^2 L^2}{c^2 R m}$. This is a differential equation for v which we can solve by separation of variables:

$$\frac{dv}{g - \alpha v} = dt$$

$$g - \alpha v(t) = (g - \alpha v(t_0)) e^{-\alpha t}$$

$v(t)$ exponentially approaches the terminal velocity $v_c = g/\alpha$ with a decay time of $1/\alpha$.

b) How much power is being dissipated by the resistor when the rail is moving at the terminal velocity? Who is doing this work?

The magnetic force is responsible for making an EMF, but it doesn't do any work:

$$\vec{F} \sim \vec{v} \times \vec{B} \perp \vec{B}.$$

The charges in the rail have a horizontal velocity \vec{u} as well as the downward velocity \vec{v} . So the magnetic force has a component quB pointing up. This is the force that gravity is countering:

$$f_{grav} = uB/c$$

downward. The particle is actually moving in the direction of the resultant velocity $\vec{w} = \vec{u} + \vec{v}$ and, because of this, in traversing the rail it goes a distance

$$L / \cos \theta > L$$

($\vec{u} \cdot \vec{w} = |u||w| \cos \theta$.) The work done in moving a unit charge once around the loop is then

$$\int f_{grav} \cdot d\vec{s} = \left(\frac{uB}{c}\right) \left(\frac{L}{\cos \theta}\right) \cos(\pi/2 - \theta) = \frac{BLu \tan \theta}{c} = \frac{BLv}{c}.$$

Note that the $\cos(\pi/2 - \theta) = \sin \theta$ came from the dot product between the gravitational force and the direction of \vec{w} . But this is exactly the EMF:

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{s} = -\frac{1}{c} \frac{\partial}{\partial t} \int B \cdot da = \frac{-BLv}{c}.$$

The power dissipated by the resistor is

$$P = \frac{\mathcal{E}^2}{R} = \frac{B^2 L^2 v^2}{c^2 R}.$$

At the terminal velocity, the rate at which the gravitational field does work is

$$\frac{\partial U}{\partial t} = F_{grav} v_c = mgv_c.$$

Plugging in our expression above for the terminal velocity, this is

$$mg \frac{mgcR}{B^2 L^2} = \frac{m^2 g^2 c^2 R}{B^2 L^2}.$$

Plugging in v_c to the expression for the power gives

$$P = \frac{B^2 L^2}{c^2 R} \left(\frac{mgc^2 R}{B^2 L^2}\right)^2 = \frac{m^2 g^2 c^2 R}{B^2 L^2}$$

which is the same.

3 varying R

What would have happened in the previous problem if the resistivity of the rail were the same as the resistivity of the 'U'?

It's very messy:

$$m\ddot{y} = F_{mag} \sim \frac{B^2 x^2 v}{c^2 (R_0 + yr)}$$

where R_0 is the resistance at $y = 0$, and r is the resistance per unit length of both arms of the 'U'. This is a nonlinear differential equation for $y, v = \dot{y}, a = \ddot{y}$.

4 Newton's third law in magnetostatics

[Griffiths 5.43] Magnetostatics treats the 'source current' ($B = \frac{I}{c} \frac{d\vec{\ell} \times \hat{r}}{r^2}$) and the 'receiving current' ($F = q\vec{v} \times B$) very asymmetrically. Let's make sure that Newton's third law holds for the force between two loops of current.

Field of loop one is given by Biot-Savart:

$$\vec{B} = \frac{I_1}{c} \oint_1 \frac{d\vec{\ell}_1 \times \hat{r}_{12}}{r_{12}^2}.$$

Lorentz force on loop two is

$$\vec{F} = \frac{I_2}{c} \oint_2 d\vec{\ell}_2 \times \vec{B} = \frac{I_1 I_2}{c^2} \oint_1 \oint_2 \frac{d\vec{\ell}_1 \times (d\vec{\ell}_2 \times \hat{r}_{12})}{r_{12}^2}.$$

Here $\vec{r}_{12} \equiv \vec{r}_1 - \vec{r}_2$. The "BAC-CAB rule" says $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$, for any three vector fields A, B, C , which in this case says

$$d\vec{\ell}_1 \times (d\vec{\ell}_2 \times \hat{r}) = d\vec{\ell}_1 (d\vec{\ell}_2 \cdot \hat{r}) - \hat{r} (d\vec{\ell}_1 \cdot d\vec{\ell}_2).$$

So

$$\vec{F} = -\frac{I_1 I_2}{c^2} \left(\oint_1 \oint_2 \frac{\hat{r}_{12}}{r_{12}^2} (d\vec{\ell}_1 \cdot d\vec{\ell}_2) - \oint_1 \oint_2 d\vec{\ell}_1 \left(\frac{d\vec{\ell}_2 \cdot \hat{r}_{12}}{r_{12}^2} \right) \right).$$

The first term here is nice and symmetric under interchange of 1 and 2. The second term is

$$\oint_1 \oint_2 d\vec{\ell}_1 (d\vec{\ell}_2 \cdot \hat{r}_{12}) = 0.$$

Here's why: This is

$$\oint_1 d\vec{\ell}_1 \left[\oint_2 (d\vec{\ell}_2 \cdot \hat{r}_{12}) \right]$$

and the stuff inside the square brackets is zero. Why? Well,

$$\frac{\hat{r}_{12}}{r_{12}^2} = -\vec{\nabla}_2 \left(\frac{1}{r_{12}} \right)$$

¹ so the thing in square brackets is

$$\left[\oint_2 d\vec{\ell}_2 \cdot \hat{r}_{12} \right] = \oint_2 \vec{\nabla}_2 \left(-\frac{1}{r_{12}} \right) \cdot d\vec{\ell}_2$$

but this is the integral over a closed loop of the divergence of a function, which by the FTC is zero.

So we have

$$\vec{F}_{12} = -\frac{I_1 I_2}{c^2} \oint_1 \oint_2 \frac{\hat{r}_{12}}{r_{12}^2} (d\vec{\ell}_1 \cdot d\vec{\ell}_2)$$

which clearly reverses sign under interchange of 1 and 2 as it should according to Sir Isaac.

¹and why is this? well, let's look at the x component of the BHS of the equation I claimed:

$$\left(\vec{\nabla}_2 \left(\frac{1}{r_{12}} \right) \right)_x = \frac{\partial}{\partial x_2} \left((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \right)^{-1/2} = -\frac{1}{2} \frac{1}{r_{12}^2} 2(x_1 - x_2) = -\frac{1}{r_{12}^2} (\hat{r}_{12})_x.$$

5 Betatron

[Griffiths 7.16] A time-varying magnetic field causes an electric field whether or not there is a wire present:

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}.$$

An electron in a uniform magnetic field goes in circles: "cyclotron motion".

$$qvB/c = \frac{mv^2}{R}$$

$$\implies qBR = mv \quad (\text{cyclotron formula}).$$

If you increase B , you speed up the electron. The acceleration is caused by the associated \vec{E} , tangential to the electron motion. It would be nice to keep the radius of the orbit constant while doing this (so you can keep the electron going around the same circle as it goes faster). A device designed to accomplish this is called a *betatron*. How should we arrange \vec{B} to make this so? Assume that \vec{B} is symmetric about the center of the orbit.

If we differentiate the BHS of the cyclotron formula wrt t , we get

$$\frac{1}{c} q \dot{B} R + \frac{1}{c} q B \dot{R} = m \dot{v} = ma$$

(q, m don't vary in time). So in order for R to stay fixed as B varies, we must have

$$\frac{q}{c} \dot{B} R = ma.$$

The acceleration of the electron satisfies

$$m\vec{a} = q\vec{E}.$$

Finally, if we call the flux through the orbit $\Phi \equiv \int \vec{E} \cdot d\vec{a}$, Faraday says

$$-\frac{1}{c} \frac{\partial \Phi}{\partial t} = \oint \vec{B} \cdot d\vec{\ell} = E \cdot 2\pi R.$$

Putting these together, we learn

$$\frac{R\dot{B}}{c} = E = -\frac{1}{2\pi Rc} \dot{\Phi}.$$

This tells us that

$$B = -\frac{1}{2\pi R^2} \Phi + \text{const.}$$

If we start with no field, then we learn that the constant must be zero, so

$$B(R) = \frac{1}{2\pi R^2} \Phi$$

which says that the field at the radius of the orbit must be *half* of the average field over the cross-section of the orbit.

6 comparison between inductors and capacitors

It's useful to make a point by point comparison between capacitors and inductors.

$$C = \frac{Q}{V}, \quad L = \frac{\mathcal{E}}{(dI/dt)}$$

In a capacitor, charge cannot change instantly; the time constant is $\tau = RC$. In an inductor, current cannot change instantly; the time constant is $\tau = L/R$.

Calculate C by assuming Q , finding V and dividing. Calculate L by assuming I , finding Φ and dividing.

A capacitor stores electrical energy,

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}.$$

We can derive this by considering how much work is done in charging up the capacitor:

$$dU = VdQ = \frac{Q}{C}dQ$$

An inductor stores magnetic energy. How much?

Note that here we don't mean the energy that gets lost through resistors as the current flows in steady state. Rather we mean the energy it costs to start the current in the first place. We can derive it by considering how much work needs to be done against the back EMF to set up the current from nothing. The work done in moving a unit charge around the circuit against the back EMF is

$$dw = -\mathcal{E} = -L\frac{dI}{dt}$$

where we used the definition of L . When the current is I , the amount of charge that moves around the circuit in time dt is $I dt$, so the work done in time dt is

$$dw = -L\frac{dI}{dt}I dt = -LI dI.$$

The total energy stored in a circuit with self-inductance L and current I is then

$$U = -\int dw = L \int I dI = \frac{1}{2}LI^2.$$

7 self-inductance of coiled donut

Suppose we have a donut that we don't want to eat. Wrap a wire around it many, say N , times. Suppose also that the radius ℓ of the donut is bigger than its thickness – say it's cross-sectional area

is A , then $\ell \gg \sqrt{A}$ (so there wouldn't be all that much to eat anyway). What's the self-inductance of this coil?

Consider an amperian loop which runs around the inside of the donut. Along the midsection, the magnetic field runs axially. If there's a current I flowing in the wire, we have

$$\oint \vec{B} \cdot d\vec{s} = B_\varphi 2\pi\ell = \frac{4\pi}{c} I_{\text{enclosed}} = \frac{4\pi IN}{c}.$$

The total magnetic flux through the wire is N times the flux through a single turn which is

$$\Phi_1 = \int \vec{B} \cdot d\vec{a}.$$

In the approximation $\ell \gg \sqrt{A}$, we can approximate B as constant over the cross-section and we have

$$\Phi = N\Phi_1 = NBA = N^2 A \frac{I}{c\ell}.$$

If we then varied the current, there would be a back EMF of size

$$|\mathcal{E}_{\text{back}}| = \frac{1}{c} \frac{\partial \Phi}{\partial t} = \frac{2N^2 A}{c^2 \ell} \frac{\partial I}{\partial t}$$

By definition of the self-inductance L , this is

$$|\mathcal{E}_{\text{back}}| = L \frac{\partial I}{\partial t}$$

so

$$L = \frac{2N^2 A}{c^2 \ell}.$$

Note that the units are indeed $\frac{cm}{s^2}$.

7.1 actually

It's possible to show that the magnetic field inside such a toroidal coil is totally axial, just using Biot-Savart. Suppose the inner radius is a and the outer radius is b . This means that if we pick an Amperian loop which is a circle of radius r , $a < r < b$ we get

$$\int \vec{B} \cdot d\vec{s} = 2\pi r B_\varphi = \frac{4\pi}{c} NI.$$

From this we learn that

$$\vec{B} = \hat{\varphi} \frac{2NI}{rc}, \text{ for } a < r < b,$$

and $\vec{B} = 0$ otherwise. This is actually true no matter what the cross-section of the donut is.

For simplicity, let's consider the case that the donut is square. Then the exact flux through the coil is

$$\Phi = N \int \vec{B} \cdot d\vec{a} = N(b-a) \int_a^b dr \frac{2NI}{rc} = 2N^2 \frac{I}{c} (b-a) \ln \frac{b}{a}.$$

The self-inductance is

$$L = 2N^2 \frac{1}{c^2} (b-a) \ln \frac{b}{a}.$$

In the limit $b-a \ll a \equiv \ell$, let's check that we get back our previous approximate answer. In this limit, we have

$$(b-a) \ln \frac{b}{a} = (b-a) \ln \left(1 + \frac{b-a}{a} \right) = \frac{(b-a)^2}{a} + \mathcal{O}\left(\frac{b-a}{a}\right) \sim \frac{A}{\ell}.$$

so we get

$$L = 2N^2 \frac{1}{c^2} \frac{A}{\ell} \left(1 + \mathcal{O}\left(\frac{b-a}{a}\right) \right),$$

which is right.

8 mutual inductance of coaxial solenoids

Take two very long solenoids which share the same axis, say \hat{z} ; the bigger one has cross-sectional area A_1 and N_1 turns and the smaller one has cross-sectional area A_2 and N_2 turns. Suppose they both have length ℓ . What are the mutual inductances M_{12} and M_{21} ?

If there's a current I_1 in the outer coil, it will create a magnetic field inside

$$\vec{B}_1 = \hat{z} \frac{4\pi N_1}{c\ell} I_1.$$

So the flux through one turn of the inner solenoid is

$$\Phi_{from\ 1\ through\ 2} = A_2 B_1 = \frac{4\pi N_1 I_1 A_2}{c\ell}$$

and the flux through the whole thing is N_2 times as big. So, if we varied I_1 , the EMF through 2 would be

$$|\mathcal{E}_{from\ 1\ through\ 2}| = \frac{4\pi N_1 N_2 A_2}{c^2 \ell} \frac{\partial I_1}{\partial t}$$

By definition, this is

$$|\mathcal{E}_{from\ 1\ through\ 2}| = M_{21} \frac{\partial I_1}{\partial t}$$

so

$$M_{21} = \frac{4\pi N_1 N_2 A_2}{c^2 \ell}$$

If instead we had a current I_2 through 2, the magnetic field would only be nonzero inside the inner coil, and it would be

$$\vec{B}_2 = \hat{z} \frac{4\pi N_2}{c\ell} I_2$$

there. Although the magnetic field is zero at the outer coil, the total magnetic flux through the outer coil is

$$\Phi_{from\ 2\ through\ 1} = N_1 A_2 B_2 = \frac{4\pi N_1 N_2 I_2 A_2}{c\ell}$$

and so the EMF in coil 1 resulting from varying I_2 is

$$|\mathcal{E}_{\text{from 1 through 2}}| = \frac{4\pi N_1 N_2 A_2}{c^2 \ell} \frac{\partial I_2}{\partial t} \equiv M_{12} \frac{\partial I_2}{\partial t}$$

and so we see that $M_{12} = M_{21}$ by what was apparently a strange series of coincidences.

But actually...

9 reciprocity of mutual inductance

(A use for the vector potential.)

Let's prove the surprising fact that $M_{12} = M_{21}$ for any two circuits. This is an opportunity for me to convey a fact that was skipped when we talked about magnetostatics. The formula for the electrostatic potential resulting from some distribution of charge is simple; in the same way, there's a simple formula for the vector potential resulting from some distribution of current.

9.1 vector potential from a steady current

\vec{A} satisfies $\vec{B} = \vec{\nabla} \times \vec{A}$. Let's stick this in the relation between B and the current:

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \frac{4\pi}{c} \vec{J}. \quad (\star)$$

We have to be careful about the BAC-CAB rule here because the derivatives act on everything to their right. The fact of the matter (which can be checked in components) is that

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}.$$

By a gauge transformation

$$\vec{A} \mapsto \vec{A} + \vec{\nabla} \chi$$

we can vary the divergence of A by

$$\vec{\nabla} \cdot \vec{A} \mapsto \vec{\nabla} \cdot \vec{A} + \nabla^2 \chi.$$

Let's choose χ to make

$$\vec{\nabla} \cdot \vec{A} = 0$$

(this is called "coulomb gauge"); we don't need to know what χ is, just that this is possible, which it is. This is convenient because then the Maxwell equation \star becomes

$$-\nabla^2 \vec{A} = \frac{4\pi}{c} \vec{J}.$$

This is nice because say the x component of it is

$$-\nabla^2 A_x = \frac{4\pi}{c} J_x.$$

But this is an equation we know how to solve from electrostatics, since it's identical in form to

$$-\nabla^2\phi = 4\pi\rho,$$

whose solution was

$$\phi(r_2) = \int d^3r_1 \frac{\rho(r_1)}{r_{21}}.$$

Making the appropriate replacements, the solution for the vector potential is

$$A_x(\vec{r}_2) = \frac{1}{c} \int d^3r_1 \frac{J_x(r_1)}{r_{21}},$$

and similarly for the other components, i.e.

$$\vec{A}(\vec{r}_2) = \frac{1}{c} \int d^3r_1 \frac{\vec{J}(r_1)}{r_{21}},$$

Having solved this we can then find \vec{B} ; if you take the curl of the BHS of this last equation, you get the Biot-Savart law back.

Let's apply this result for a thin wire. Then

$$\vec{J}d^3r_1 = Id\vec{s}_1.$$

The vector potential at a point \vec{r}_2 resulting from a current I_2 flowing along a path C_1 is given by:

$$\vec{A}(\vec{r}_2) = \frac{I}{c} \oint_{C_1} \frac{d\vec{s}_1}{r_{21}}, \quad (1)$$

where C_1 is the path of the wire.

9.2 back to mutual inductance

The vector potential created by a current I in circuit one is

$$\vec{A}_{21} = \frac{I}{c} \int_{C_1} \frac{d\vec{s}_1}{r_{21}}.$$

The flux through loop two is

$$\Phi_{21} = \int_2 \vec{B}_{21} \cdot d\vec{a}_2 = \int_2 \vec{\nabla} \times \vec{A}_{21} \cdot d\vec{a}_2 \stackrel{\text{Stokes}}{=} \int_{C_2} \vec{A}_{21} \cdot d\vec{s}_2.$$

Plugging in our expression for A_{21} , this is

$$\Phi_{21} = \oint_{C_2} d\vec{s}_2 \cdot \left(\frac{I}{c} \oint_{C_1} \frac{d\vec{s}_1}{r_{21}} \right) = \frac{I}{c} \oint_{C_2} \oint_{C_1} \frac{d\vec{s}_1 \cdot d\vec{s}_2}{r_{21}}.$$

But this is exactly what we would have found for the flux Φ_{12} through C_1 due to a current I in C_2 :

$$\Phi_{21} = \Phi_{12}.$$

This means that

$$M_{21} = \frac{\partial\Phi_{21}}{\partial I} = \frac{\partial\Phi_{12}}{\partial I} = M_{12}.$$

So in a given problem, you can pick whichever one M_{12} or M_{21} is easier to calculate.

9.3 an application

Suppose that in the problem with the coaxial solenoids the inner solenoid were not only narrower than the outer one, but also *shorter*, i.e. they had lengths $\ell_1 < \ell_2$. In this case, a current through the inner solenoid would produce a complicated fringing field and it would be hard to calculate M_{12} . M_{21} however is actually the same, if we set $\ell = \ell_1$ and if the total number of coils N_2 is the same. By the reciprocity theorem, this means that M_{21} is the same, too. Somehow.

10 minimizing self-inductance

[Purcell 7.8] How would you wind a coil (which you want to be long to get a large resistance, say) so that the self-inductance is small?

You want to minimize the magnetic field. Here's a good trick: first take the wire, line up the two ends and let the rest hang down, so the wire is folded back on itself (make sure it's coated with an insulator!). Now take this and wind it around something. The current will run in opposite directions on neighboring segments and so the resulting magnetic field (the superposition of these) will be small.

11 magnetic energy

I said that our expression for the energy stored in an inductor described energy stored in the magnetic field. To make this more precise, I claim that our expression is equal to

$$U_{\text{magnetic}} = \frac{1}{2}LI^2 = \frac{1}{8\pi^2} \int_{\text{all space}} B^2 d^3r,$$

and that the latter expression is much more general.

11.1

Here's a case where this gives a better way to find L : Suppose we have coaxial cables of radius a and b with a current I flowing along the outer one and back along the inner one. Ampere tells us that the field is only nonzero in between the cylinders and is

$$\vec{B}(r) = \hat{\phi} \frac{2I}{rc}$$

(note that this is actually a kind of toroidal coil).

The energy per unit volume is

$$\frac{B^2}{8\pi^2} = \frac{4I^2}{8\pi^2 r^2 c^2}$$

and the volume integral of this, if the cable has length ℓ is

$$U = \int_a^b r dr \int_0^\ell dz \int_0^{2\pi} d\varphi \frac{4I^2}{8\pi^2 r^2 c^2} = 2\pi\ell \frac{I^2}{2\pi^2 c^2} \ln \frac{b}{a} = \frac{\ell}{\pi c^2} \ln \frac{b}{a} I^2.$$

This must be $\frac{1}{2}LI^2$ so we have

$$L = \frac{2\ell}{\pi c^2} \ln \frac{b}{a}.$$