MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
Physics II (8.022) - Prof. J. McGreevy - Spring 2008

Problem Set 1 **Solutions**

TA - Nan Gu

1. Taylor Expansion [10 points]

We first make the substitution $u = \frac{1}{x}$ and simplify the resulting equation. We also make use of the properties of log to express the result as a sum of logs of polynomials:

$$f(u) = \ln\left(\frac{a_1 + \frac{1}{u}}{\frac{1}{u}}\right) + \ln\left(\frac{a_2 + \frac{1}{u}}{a_1 + a_2 + \frac{1}{u}}\right) = \ln(1 + a_1 u) + \ln(1 + a_2 u) - \ln(1 + a_1 u + a_2 u)$$

Recall that the Taylor series for a function $g(u)$ around the point $u = 0$ is:

$$g(u) = g(0) + \frac{d}{du}g(u)\bigg|_{u=0} u + \frac{1}{2}\frac{d^2}{du^2}g(u)\bigg|_{u=0} u^2 + ...$$

Specializing to natural log:

$$\ln(1 + x) = 0 + x - \frac{x^2}{2} + ...$$

Using the second order expansion for $\ln(1 + x)$ on our $f(u)$ gives:

$$f(u) \approx \left(a_1 u - \frac{a_1^2 u^2}{2}\right) + \left(a_2 u - \frac{a_2^2 u^2}{2}\right) - \left((a_1 + a_2)u - \frac{(a_1 + a_2)^2 u^2}{2}\right)$$

Simplifying:

$$f(u) \approx a_1 a_2 u^2 \rightarrow f(x) \approx \frac{a_1 a_2}{x^2}$$

2. The gradient in various coordinate systems [20 points]

The three coordinate systems that we will consider use the following coordinates and have the following relations:

Cartesian: x, y, z

Cylindrical: $R = \sqrt{x^2 + y^2}$, $\varphi = \arctan\frac{y}{x}$, $z = z$; $(x = R\cos\varphi,\ y = R\sin\varphi,\ z = z)$

Spherical: $r = \sqrt{x^2 + y^2 + z^2}$, $\varphi = \arctan\frac{y}{x}$, $\theta = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}}$; $(x = r\cos\varphi\sin\theta,\ y = r\sin\varphi\sin\theta,\ z = r\cos\theta)$

(a) position vector

Let the vector from the origin to the point P be denoted as $\vec{r}$.

Cartesian: $\vec{r} = x\ \hat{x} + y\ \hat{y} + z\ \hat{z}$

Cylindrical: $\vec{r} = R\cos\varphi\ \hat{x} + R\sin\varphi\ \hat{y} + z\ \hat{z} = R\ \hat{R} + z\ \hat{z}$

Spherical: $\vec{r} = r\cos\varphi\sin\theta\ \hat{x} + r\sin\varphi\sin\theta\ \hat{y} + r\cos\theta\ \hat{z} = r\ \hat{r}$

(b) line element

Cartesian: $d\vec{r} = dx\ \hat{x} + dy\ \hat{y} + dz\ \hat{z}$

Cylindrical: $d\vec{r} = dR\ \hat{R} + Rd\varphi\ \hat{\varphi} + dz\ \hat{z}$

Spherical: $d\vec{r} = dr\ \hat{r} + r\sin\theta d\varphi\ \hat{\varphi} + rd\theta\ \hat{\theta}$

(c) volume element

Cartesian: $dV = dxdydz$
Cylindrical: $dV = RdRd\varphi dz$
Spherical: $dV = r^2 \sin\theta dr d\varphi d\theta$

(d) gradient

Cartesian: f is a function of x, y, and z
$\vec{\nabla} f(x,y,z) = A\ \hat{x} + B\ \hat{y} + C\ \hat{z} \qquad d\vec{r} = dx\ \hat{x} + dy\ \hat{y} + dz\ \hat{z}$
$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \vec{\nabla} f \cdot d\vec{r} \qquad \rightarrow \qquad A = \frac{\partial f}{\partial x} \qquad B = \frac{\partial f}{\partial y} \qquad C = \frac{\partial f}{\partial z}$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\ \hat{x} + \frac{\partial f}{\partial y}\ \hat{y} + \frac{\partial f}{\partial z}\ \hat{z}$$

Cylindrical: f is a function of R, φ, and z
$\vec{\nabla} f(R,\varphi,z) = a\ \hat{R} + b\ \hat{\varphi} + c\ \hat{z} \qquad d\vec{r} = dR\ \hat{R} + Rd\varphi\ \hat{\varphi} + dz\ \hat{z}$
$df = \frac{\partial f}{\partial R}dR + \frac{\partial f}{\partial \varphi}d\varphi + \frac{\partial f}{\partial z}dz = \vec{\nabla} f \cdot d\vec{r} \qquad \rightarrow \qquad a = \frac{\partial f}{\partial R} \qquad b = \frac{1}{R}\frac{\partial f}{\partial \varphi} \qquad c = \frac{\partial f}{\partial z}$

$$\vec{\nabla} f = \frac{\partial f}{\partial R}\ \hat{R} + \frac{1}{R}\frac{\partial f}{\partial \varphi}\ \hat{\varphi} + \frac{\partial f}{\partial z}\ \hat{z}$$

Spherical: f is a function of r, φ, and θ
$\vec{\nabla} f(r,\varphi,\theta) = \alpha\ \hat{r} + \beta\ \hat{\varphi} + \gamma\ \hat{\theta} \qquad d\vec{r} = dr\ \hat{r} + r\sin\theta d\varphi\ \hat{\varphi} + rd\theta\ \hat{\theta}$
$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \varphi}d\varphi + \frac{\partial f}{\partial \theta}d\theta = \vec{\nabla} f \cdot d\vec{r} \qquad \rightarrow \qquad \alpha = \frac{\partial f}{\partial r} \qquad \beta = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi} \qquad \gamma = \frac{1}{r}\frac{\partial f}{\partial \theta}$

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\ \hat{r} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\ \hat{\varphi} + \frac{1}{r}\frac{\partial f}{\partial \theta}\ \hat{\theta}$$

3. Checking your answer to problem two [10 points]

(a) $f(x,y,z) = y$

Cartesian: $f(x,y,z) = y$

$$\vec{\nabla} f(x,y,z) = \hat{y}$$

Cylindrical: $f(R,\varphi,z) = R\sin\varphi$

$$\vec{\nabla} f(R,\varphi,z) = \sin\varphi\ \hat{R} + \cos\varphi\ \hat{\varphi}$$

To compare the two, we need to convert between Cylindrical and Cartesian unit vectors. We use the definition of a unit vector along any vector $\vec{d}$ to compute $\hat{R}$ and $\hat{\varphi}$ (where $\vec{r} = x\ \hat{x} + y\ \hat{y} + z\ \hat{z}$ is the position vector):

$$\hat{d} = \frac{\partial \vec{r}}{\partial d}\Big/\left|\frac{\partial \vec{r}}{\partial d}\right| \qquad \rightarrow \qquad \begin{aligned} \hat{R} &= \cos\varphi\ \hat{x} + \sin\varphi\ \hat{y} \\ &= \tfrac{1}{\sqrt{x^2+y^2}}(x\ \hat{x} + y\ \hat{y}) \end{aligned} \qquad \begin{aligned} \hat{\varphi} &= -\sin\varphi\ \hat{x} + \cos\varphi\ \hat{y} \\ &= \tfrac{1}{\sqrt{x^2+y^2}}(-y\ \hat{x} + x\ \hat{y}) \end{aligned}$$

So we see that: $\hat{y} = \sin\varphi\ \hat{R} + \cos\varphi\ \hat{\varphi}$, demonstrating the equivalence of our two expressions for $\vec{\nabla} f$.

(b) $r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

Cartesian: $r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$$\vec{\nabla} r(x, y, z) = \frac{x\ \hat{x} + y\ \hat{y} + z\ \hat{z}}{\sqrt{x^2 + y^2 + z^2}}$$

Spherical: $r(r, \varphi, \theta) = r$

$$\vec{\nabla} r(r, \varphi, \theta) = \hat{r}$$

We can find $\hat{r}$ by the same method as in part **(a)**

$$\hat{r} = \cos\varphi \sin\theta\ \hat{x} + \sin\varphi \sin\theta\ \hat{y} + \cos\theta\ \hat{z} = \frac{x\ \hat{x} + y\ \hat{y} + z\ \hat{z}}{\sqrt{x^2 + y^2 + z^2}}$$

4. Purcell 1.1 Comparison of electric and gravitational forces [10 points]

Following Purcell, we will use CGS units for this problem.
Comparing the magnitudes of the gravitational and electrostatic forces between two objects of masses and charges m_1, q_1 and m_2, q_2 separated by a distance r:

$$|F_G| = \frac{Gm_1m_2}{r^2} \qquad |F_E| = \frac{kq_1q_2}{r^2} \qquad \rightarrow \qquad \frac{|F_G|}{|F_E|} = \frac{Gm_1m_2}{kq_1q_2}$$

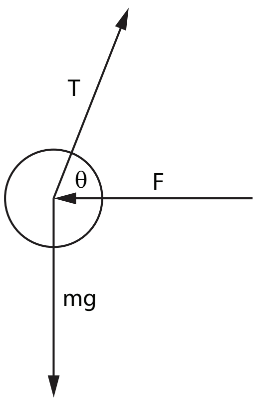
We calculate this quantity between two protons ($m_1 = m_2 = m_p = 1.6 \times 10^{-24}\ gm$, $q_1 = q_2 = e = 4.8 \times 10^{-10}\ esu$, $k = 1$):

$$\frac{|F_G|}{|F_E|} = \frac{Gm_p^2}{ke^2} = \frac{6.7 \times 10^{-8}(1.6 \times 10^{-24})^2}{(4.8 \times 10^{-10})^2} = 7.44 \times 10^{-37}$$

So we see that between elementary particles, the gravitational force is utterly negligible. The force of repulsion between two protons in a Helium nucleus is:

$$|F_E| = \frac{(4.8 \times 10^{-10})^2}{(10^{-13})^2} = 2.3 \times 10^7\ dynes = 230\ N = 51.7\ lb$$

5. Purcell 1.3 Two charged volleyballs [10 points]



Following Purcell, we will use SI units for this problem.
Since the problem is symmetric for the two volleyballs, the equilibrium condition for both volleyballs are the same. Using a force diagram gives the following equations:

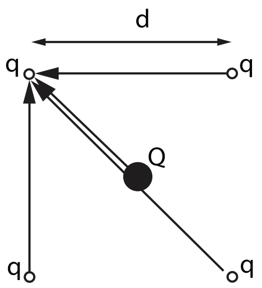
$$T\cos\theta = F = \frac{kq^2}{x^2} \qquad T\sin\theta = mg \qquad \rightarrow \qquad \cot\theta = \frac{kq^2}{mgx^2}$$

where m and q are the mass and charge of each ball and x their separation. $\cot\theta = \frac{x}{2y}$ where y is the vertical distance to the ceiling. With all this, we find:

$$q^2 = \frac{mgx^3}{k2y} = \frac{0.3 \times 9.8 \times (0.5)^3}{9 \times 10^9 \times 2 \times 2.5} \rightarrow q = 2.86 \times 10^{-6}\ C$$

6. Purcell 1.4 Charges on the corners of a square [10 points]

No explicit choice of units is necessary, but we will use the CGS expression for the Coulomb force.



This problem is also symmetric for the four charges, so we will calculate the equilibrium condition for a single charge, say, the top left one. Recall that the condition for equilibrium is: $\sum \vec{F} = 0$

$$\sum \vec{F} = -\frac{q^2}{d^2}\,\hat{x} + \frac{q^2}{d^2}\,\hat{y} + \frac{q^2}{(\sqrt{2}d)^2}\left(-\frac{\hat{x}}{\sqrt{2}} + \frac{\hat{y}}{\sqrt{2}}\right) + \frac{Qq}{(d/\sqrt{2})^2}\left(-\frac{\hat{x}}{\sqrt{2}} + \frac{\hat{y}}{\sqrt{2}}\right) = 0$$

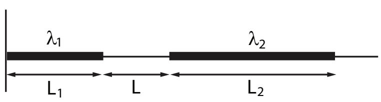
Recall also that each component of a vector equation must be independently satisfied. Let's take the $\hat{x}$-component equation:

$$-\frac{q^2}{d^2} - \frac{q^2}{2\sqrt{2}d^2} - \frac{2Qq}{\sqrt{2}d^2} = 0 \rightarrow Q = -\frac{2\sqrt{2}+1}{4}q = -0.957q$$

This system is not in a stable equilibrium. Imagine moving the central charge slightly in any direction. The corner charges that are now closer will move towards the central charge, while the corner charges that are further will move away. In fact, there is a theorem in electrostatics called *Earnshaw's Theorem* that states there can be no configuration of point charges can be stable if the only forces involved are the electrostatic forces of those charges.

7. Coulomb force between line charges [20 points]

(a)



The total force on each rod is just the integral of all the infinitesimal forces:

$$\vec{F} = \hat{F}\int_0^{L_1} dx_1 \int_{L_1+L}^{L_1+L+L_2} dx_2 \frac{\lambda_1\lambda_2}{(x_2-x_1)^2}$$

where both x_1 and x_2 are measured from the left end of the left rod. Since this is a one-dimensional problem, $\hat{F}$ is either $+\hat{x}$ or $-\hat{x}$ depending on whether we are considering the force from the left rod to the right rod or vice versa.

Completing the integral:

$$F = -\lambda_1\lambda_2\int_0^{L_1} dx_1\left[\frac{1}{x_2-x_1}\right]_{L_1+L}^{L_1+L+L_2} = -\lambda_1\lambda_2\int_0^{L_1} dx_1\left(\frac{1}{L_1+L+L_2-x_1} - \frac{1}{L_1+L-x_1}\right)$$

$$F = -\lambda_1\lambda_2\left[\log|L_1+L+L_2-x_1| - \log|L_1+L-x_1|\right]_0^{L_1}$$

$$F = -\lambda_1\lambda_2\left(\log|L+L_2| - log|L_1+L+L_2| - \log|L| + \log|L_1+L|\right) = -\lambda_1\lambda_2\log\left(\frac{(L+L_2)(L+L_1)}{L(L_1+L+L_2)}\right)$$

(b)

By splitting the log up as follows, we isolate the relevant term:

$$\log\left(\frac{(L+L_2)(L+L_1)}{L(L_1+L+L_2)}\right) = \log\left(\frac{L+L_1}{L}\right) + \log\left(\frac{L+L_2}{L+L_1+L_2}\right)$$

The second term is nearly zero as $L_2 \gg L_1$ and so we are only left with the first term. (Formally, the second term is $-\log(1+\frac{L_1}{L+L_2}) \approx -\frac{L_1}{L+L_2}$ which is first order in $\frac{L_1}{L_2}$ for positive L.) The first term is exactly what we need.
Now we restore the fact that the force is from rod 2 acting on rod 1 and must have a negative direction:

$$\vec{F} = -\hat{x}\lambda_1\lambda_2 \log\left(1+\frac{L_1}{L}\right)$$

(c)

In the limit where $L \gg L_1, L_2$ we have the exact same equation as we did for problem 1.

$$\log\left(\frac{(L+L_2)(L+L_1)}{L(L_1+L+L_2)}\right) = \log\left(\frac{L_1+L}{L}\right) + \log\left(\frac{L+L_2}{L_1+L_2+L}\right)$$

Hence, the Coulomb force is simplified back to the original equation for point charges:

$$\vec{F} = \hat{x}\frac{\lambda_1\lambda_2 L_1 L_2}{L^2} = \hat{x}\frac{Q_1 Q_2}{L^2}$$