Displacement Current, Maxwell’s Equations, Wave Equations

Maxwell’s Equations

\[ \oint_S \mathbf{E} \cdot \mathbf{n} \, da = \frac{1}{\varepsilon_0} \iint_S \rho \, dV \]  (Gauss’s Law)

\[ \oint_S \mathbf{B} \cdot \mathbf{n} \, da = 0 \]  (Magnetic Gauss’s Law)

\[ \oint_C \mathbf{E} \cdot \mathbf{d}s = \frac{d}{dt} \iint_S \mathbf{B} \cdot \mathbf{n} \, da \]  (Faraday’s Law)

\[ \oint_C \mathbf{B} \cdot \mathbf{d}s = \mu_0 \iint_S \mathbf{J} \cdot \mathbf{n} \, da \]  (Ampere’s Law quasi-static)

Is there something missing?

Maxwell’s Equations
One Last Modification: Displacement Current
Ampere’s Law: Capacitor

Consider a charging capacitor:

Use Ampere’s Law to calculate the magnetic field just above the top plate

\[ \oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{\text{enc}} \]

\[ I_{\text{enc}} = \int_S \mathbf{J} \cdot \hat{n} \, da \]

1) Surface \( S_1 \): \( I_{\text{enc}} = I \)
2) Surface \( S_2 \): \( I_{\text{enc}} = 0 \)

What’s Going On?

Displacement Current

We don’t have current between the capacitor plates but we do have a changing \( E \) field. Can we “make” a current out of that?

\[ E = \frac{Q}{\varepsilon_0 A} \Rightarrow Q = \varepsilon_0 EA = \varepsilon_0 \Phi_E \]

\[ \frac{dQ}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt} = I_{\text{dis}} \]

This is called the “displacement current”. It is not a flow of charge but proportional to changing electric flux

Displacement Current:

\[ I_{\text{dis}} = \varepsilon_0 \frac{d}{dt} \int_S \mathbf{E} \cdot \hat{n} \, da = \varepsilon_0 \frac{d\Phi_E}{dt} \]

If surface \( S_2 \) encloses all of the electric flux due to the charged plate then \( I_{\text{dis}} = I \)
Maxwell-Ampere’s Law

\[ \oint_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 \oint_S \mathbf{J} \cdot \mathbf{n} \, da + \mu_0 \varepsilon_0 \frac{d}{dt} \oint_S \mathbf{E} \cdot \mathbf{n} \, da \]

\[ = \mu_0 (I_{\text{exc}} + I_{\text{dis}}) \]

“flow of electric charge”

\[ I_{\text{exc}} = \oint_S \mathbf{J} \cdot \mathbf{n} \, da \]

“changing electric flux”

\[ I_{\text{dis}} = \varepsilon_0 \frac{d}{dt} \oint_S \mathbf{E} \cdot \mathbf{n} \, da \]

Concept Question: Capacitor

If instead of integrating the magnetic field around the pictured Amperian circular loop of radius \( r \) we were to integrate around an Amperian loop of the same radius \( R \) as the plates (b) then the integral of the magnetic field around the closed path would be

1. the same.
2. larger.
3. smaller.

Sign Conventions: Right Hand Rule

Integration direction clockwise for line integral requires that unit normal points into page for surface integral.

Current positive into the page. Negative out of page.

Electric flux positive into page, negative out of page.
**Sign Conventions: Right Hand Rule**

\[
\oint_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 \oint_S \mathbf{J} \cdot \hat{n} \, da + \mu_0 \varepsilon_0 \frac{d}{dt} \oint_S \mathbf{E} \cdot \hat{n} \, da
\]

Integration direction counter clockwise for line integral requires that unit normal points out page for surface integral.

Current positive out of page.
Negative into page.

Electric flux positive out of page, negative into page.

**Concept Question: Capacitor**

Consider a circular capacitor, with an Amperian circular loop (radius \( r \)) in the plane midway between the plates. When the capacitor is charging, the line integral of the magnetic field around the circle (in direction shown) is

1. Zero (No current through loop)
2. Positive
3. Negative
4. Can’t tell (need to know direction of E)

**Concept Question: Capacitor**

The figures above show a side and top view of a capacitor with charge \( Q \) and electric and magnetic fields \( E \) and \( B \) at time \( t \). At this time the charge \( Q \) is:

1. Increasing in time
2. Constant in time.
3. Decreasing in time.
Group Problem: Capacitor

A circular capacitor of spacing $d$ and radius $R$ is in a circuit carrying the steady current $i$ shown. At time $t = 0$, the plates are uncharged.

1. Find the electric field $E(t)$ at $P$ vs. time $t$ (mag. & dir.)
2. Find the magnetic field $B(t)$ at $P$

Maxwell’s Equations

\[
\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} \int_V \rho dV \quad \text{(Gauss’s Law)}
\]

\[
\oint_S \mathbf{B} \cdot d\mathbf{a} = 0 \quad \text{(Magnetic Gauss’s Law)}
\]

\[
\oint_{\partial V} \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int_V \mathbf{B} \cdot d\mathbf{a} \quad \text{(Faraday’s Law)}
\]

\[
\oint_{\partial V} \mathbf{B} \cdot d\mathbf{s} = \mu_0 \int_{\partial V} \mathbf{E} \cdot d\mathbf{a} + \mu_0 \varepsilon_0 \frac{d}{dt} \int_V \mathbf{E} \cdot d\mathbf{a} \quad \text{(Maxwell - Ampere’s Law)}
\]

Electromagnetism Review

**E fields are associated with:**

1. electric charges \hspace{1cm} (Gauss’s Law)
2. time changing B fields \hspace{1cm} (Faraday’s Law)

**B fields are associated with**

3a) moving electric charges \hspace{1cm} (Ampere-Maxwell Law)
3b) time changing E fields \hspace{1cm} (Maxwell’s Addition (Ampere-Maxwell Law)

**Conservation of magnetic flux**

4) No magnetic charge \hspace{1cm} (Gauss’s Law for Magnetism)
**Electromagnetism Review**

Conservation of charge:
\[
\oint_{\text{closed}} \mathbf{J} \cdot d\mathbf{A} = -\frac{d}{dt} \int_{\text{enclosed}} \rho dV
\]

E and B fields exert forces on (moving) electric charges:
\[
\mathbf{F}_q = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})
\]

Energy stored in electric and magnetic fields
\[
U_E = \iiint_{\text{all space}} u_E dV = \iiint_{\text{all space}} \frac{\varepsilon_0}{2} \mathbf{E}^2 dV
\]
\[
U_B = \iiint_{\text{all space}} u_B dV = \iiint_{\text{all space}} \frac{1}{2\mu_0} \mathbf{B}^2 dV
\]

**Maxwell’s Equations in Vacua**

\[
\frac{d}{dt} \int_{\text{closed}} \mathbf{E} \cdot d\mathbf{A} = \frac{\partial}{\partial t} \int_{\text{volume}} \rho dV \quad \text{(Gauss's Law)}
\]
\[
\oint_{\text{closed}} \mathbf{B} \cdot d\mathbf{A} = 0 \quad \text{(Magnetic Gauss's Law)}
\]
\[
\oint_{\text{closed}} \mathbf{E} \cdot d\mathbf{l} = \frac{d\Phi_B}{dt} \quad \text{(Faraday's Law)}
\]
\[
\oint_{\text{closed}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enclosed}} + \mu_0 \int_{\text{volume}} \frac{d\Phi_E}{dt} \quad \text{(Ampere - Maxwell Law)}
\]

What about free space (no charge or current)?
How Do Maxwell’s Equations Lead to EM Waves?

Wave Equation
Start with Ampere-Maxwell Eq and closed oriented loop
\[ \oint_{C} \mathbf{B} \cdot d\mathbf{s} = \mu_{0} \varepsilon_{0} \frac{d}{dt} \int \mathbf{E} \cdot \hat{n} da \]

Wave Equation
Start with Ampere-Maxwell Eq:
\[ \oint_{C} \mathbf{B} \cdot d\mathbf{s} = \mu_{0} \varepsilon_{0} \frac{d}{dt} \left( \int \mathbf{E} \cdot \hat{n} da \right) \]

Apply it to red rectangle:
\[ \oint_{C} \mathbf{B} \cdot d\mathbf{s} = B_{z}(x,t)l - B_{z}(x + \Delta x,t)l \]
\[ \mu_{0} \varepsilon_{0} \frac{d}{dt} \left( \int \mathbf{E} \cdot \hat{n} da \right) = \mu_{0} \varepsilon_{0} \left( \frac{\partial E_{y}(x + \Delta x / 2,t)}{\partial t} \right) \]

So in the limit that \( dx \) is very small:
\[ \frac{\partial B}{\partial x} = \mu_{0} \varepsilon_{0} \frac{\partial E}{\partial t} \]
Group Problem: Wave Equation

Use Faraday’s Law: \( \oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{a} \)
and apply it to red rectangle to find the partial differential equation in order to find a relationship between \( \frac{\partial E_y}{\partial x} \) and \( \frac{\partial B_z}{\partial t} \).

\[
\frac{\partial E_y}{\partial x} \quad \text{and} \quad \frac{\partial B_z}{\partial t}
\]

Group Problem: Wave Equation Sol.

Use Faraday’s Law: \( \oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{a} \)
and apply it to red rectangle:

\[
\oint \mathbf{E} \cdot d\mathbf{s} = E_y(x + \Delta x, t) - E_y(x, t)
\]

\[
-\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{a} = -\Delta x \frac{\partial B_z}{\partial t}
\]

\[
E_y(x + dx, t) - E_y(x, t)
\]

So, in the limit that \( dx \) is very small:

\[
\frac{\partial E_y}{\partial x} = \frac{\partial B_z}{\partial t}
\]

1D Wave Equation for Electric Field

\[
\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \quad (1)
\]

\[
-\frac{\partial B_z}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \quad (2)
\]

Take \( x \)-derivative of Eq.(1) and use the Eq. (2)

\[
\frac{\partial^2 E_y}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial E_y}{\partial x} \right) = \frac{\partial}{\partial x} \left( -\frac{\partial B_z}{\partial t} \right) = -\frac{\partial}{\partial t} \left( \frac{\partial B_z}{\partial x} \right) = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}
\]

\[
\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}
\]
1D Wave Equation for $E$

\[
\frac{\partial^2 E_y}{\partial x^2} - \frac{\partial^2 E_y}{\partial t^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}
\]

This is an equation for a wave. Let $E_y = f(x - vt)$

\[
\begin{align*}
\frac{\partial^2 E_y}{\partial x^2} &= f''(x - vt) \\
\frac{\partial^2 E_y}{\partial t^2} &= v^2 f''(x - vt)
\end{align*}
\]

$\Rightarrow v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

Definition of Constants and Wave Speed

Recall exact definitions of

\[
c \equiv 299792458 \text{ m} \cdot \text{s}^{-1}
\]

\[
\mu_0 \equiv 4\pi \times 10^{-9} \text{ N} \cdot \text{s}^2 \cdot \text{C}^{-2}
\]

The permittivity of free space $\epsilon_0$ is exactly defined by

\[
\epsilon_0 \equiv \frac{1}{c^2 \mu_0} \equiv 8.9875517873681764 \times 10^9 \text{ C}^2 \cdot \text{m}^{-2} \cdot \text{N}^{-1}
\]

$\Rightarrow v = c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ in vacua

Group Problem: 1D Wave Eq. for $B$

\[
\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x}, \quad \frac{\partial B_z}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial E_y}{\partial t}
\]

Take appropriate derivatives of the above equations and show that

\[
\frac{\partial^2 B_z}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 B_z}{\partial t^2}
\]
Wave Equations: Summary

Both electric & magnetic fields travel like waves:

\[
\frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} \quad \frac{\partial^2 B_z}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 B_z}{\partial t^2}
\]

with speed

\[
c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}
\]

But there are strict relations between them:

\[
\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \quad \frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}
\]

Electromagnetic Radiation: Plane Waves

http://youtu.be/3ivZf_LKezc