Problem Solving 4: Capacitance, Stored Energy, Capacitors in Parallel and Series, Dielectrics

Section ______ Table __________________________

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Hand in one copy per group at the end of the Friday Problem Solving Session.

OBJECTIVES

1. To calculate the capacitance of a simple capacitor.

2. To calculate the energy stored in a capacitor in two ways.

REFERENCE: Course Notes: Sections 5.1-5.6, 5.8-5.9

PROBLEM SOLVING STRATEGIES (see Section 5.8, 8.02 Course Notes)

(1) Using Gauss’s Law, calculate the electric field everywhere.

(2) Compute the electric potential difference $\Delta V$ between the two conductors.

(3) Calculate the capacitance $C$ using $C = \frac{Q}{|\Delta V|}$. 
Problem 1: The Cylindrical Capacitor:

Two concentric conducting cylinders have radii \( a \) and \( b \) and height \( l \), with \( b > a \). The inner cylinder carries total charge \(-Q\), and the outer cylinder carries total charge \( Q \). We ignore end effects. The goal of this problem is to calculate the capacitance.

Question 1: The Electric Field

Use Gauss’s Law to find the direction and magnitude of the electric field in the between the inner and outer cylinders \((a < r < b)\). NOTE: The inner cylinder has negative charge \(-Q\).

Answer:

Question 2: Electric Potential Difference (Voltage Difference)

The voltage difference between the cylinders, \( \Delta V \), is defined to be the work done per test charge against the electric field in moving a test charge \( q \), from the inner cylinder to the outer cylinder

\[
\Delta V \equiv V(b) - V(a) = -\int_{a}^{b} \mathbf{E} \cdot d\mathbf{s}.
\]

Find an expression for the voltage difference between the cylinders in terms of the charge \( Q \), the radii \( a \) and \( b \), the height \( l \), and any other constants that you may find necessary.
Question 3: Calculating Capacitance

Our two conducting cylinders form a capacitor. The magnitude of the charge, $|Q|$, on either cylinder is related to the magnitude of the voltage difference between the cylinders according to $|Q| = C|\Delta V|$ where $\Delta V$ is the voltage difference across the capacitor and $C$ is the constant of proportionality called the ‘capacitance’. The capacitance is determined by the geometrical properties of the two conductors and is therefore independent of the applied voltage difference across the cylinders.

What is the capacitance $C$ of our system of two cylinders? Express your answer in terms, $a$, and $b$, the height $l$, and any other constants which you may find necessary.

Answer:

Question 4: Stored Electrostatic Energy

The total electrostatic energy stored in the electric fields is given by the expression,

$$U = \frac{\varepsilon_0}{2} \int_{\text{all space}} \vec{E} \cdot \vec{E} \, dV_{\text{vol}}.$$

Starting from your expression for $E$ in question (1), calculate this electrostatic energy and express your answer in terms of $Q$, $a$, $b$, and $l$ (and any other constants which you may find necessary). If you use your expression for $C$ from question 3 above, can you write you expression in terms of $Q$ and $C$ alone? What is that expression?
Answer:
Question 5: Charging the Capacitor

Suppose instead of using a battery we charge the capacitor ourselves in the following way. We move charge from the inside of the cylinder at $r = b$ to the surface of the cylinder at $r = a$. Suppose we start off with zero charge on the conductors and we move charge for awhile until at time $t$ we have built up a change $q(t)$ on the inner cylinder.

a. What is the voltage difference between the two cylinders at time $t$, in terms of $C$ and $q(t)$?

Answer:

b. Now we move a very little additional charge $dq$ from the outer to the inner cylinder. How much work $dW$ do we have to do to move that $dq$ from the outer to the inner cylinder, in the presence of the charge $q(t)$ already there, in terms of $C$, $q(t)$, and $dq$?

Answer:

c. Using your result in (b), calculate the total work we have to do to bring a total charge $Q$ from the outer to the inner cylinder, assuming the cylinders start out uncharged (Hint: integrate with respect to $dq$ from 0 to $Q$).

Answer:

d. Is the work we did in charging the capacitor greater than, equal to, or less than the stored electrostatic energy in the capacitor that you calculated in question 4? Why?

Answer:
Parallel Connection

Suppose we have two capacitors \( C_1 \) with charge \( Q_1 \) and \( C_2 \) with charge \( Q_2 \) that are connected in parallel, as shown in Figure 4.1.

![Figure 4.1 Capacitors in parallel and an equivalent capacitor.](image)

The left plates of both capacitors \( C_1 \) and \( C_2 \) are connected to the positive terminal of the battery and have the same electric potential as the positive terminal. Similarly, both right plates are negatively charged and have the same potential as the negative terminal. Thus, the potential difference \(|\Delta V|\) is the same across each capacitor. This gives

\[
C_1 = \frac{Q_1}{|\Delta V|}, \quad C_2 = \frac{Q_2}{|\Delta V|}
\]

These two capacitors can be replaced by a single equivalent capacitor \( C_{eq} \) with a total charge \( Q \) supplied by the battery. However, since \( Q \) is shared by the two capacitors, we must have

\[
Q = Q_1 + Q_2 = C_1 |\Delta V| + C_2 |\Delta V| = (C_1 + C_2) |\Delta V|
\]

By definition the equivalent capacitance is given by

\[
C_{eq} \equiv \frac{Q}{|\Delta V|}
\]

We can now substitute Eq. (2) into Eq. (3) and find that
Thus, capacitors that are connected in parallel add. The generalization to any number of capacitors is

\[ C_{\text{eq}} = C_1 + C_2 + C_3 + \cdots + C_N = \sum_{i=1}^{N} C_i \]  

(parallel)  

Series Connection

Suppose two initially uncharged capacitors \( C_1 \) and \( C_2 \) are connected in series, as shown in Figure 4.2. A potential difference \( |\Delta V| \) is then applied across both capacitors. The left plate of capacitor 1 is connected to the positive terminal of the battery and becomes positively charged with a charge \( +Q \), while the right plate of capacitor 2 is connected to the negative terminal and becomes negatively charged with charge \( -Q \) as electrons flow in. What about the inner plates? They were initially uncharged; now the outside plates each attract an equal and opposite charge. So the right plate of capacitor 1 will acquire a charge \( -Q \) and the left plate of capacitor \( +Q \).

The potential differences across capacitors \( C_1 \) and \( C_2 \) are

\[ |\Delta V_1| = \frac{Q}{C_1}, \quad |\Delta V_2| = \frac{Q}{C_2} \]  

respectively. From Figure 4.2, we see that the total potential difference is simply the sum of the two individual potential differences:

\[ |\Delta V| = |\Delta V_1| + |\Delta V_2| \]  

In fact, the total potential difference across any number of capacitors in series connection is equal to the sum of potential differences across the individual capacitors. These two
capacitors can be replaced by a single equivalent capacitor $C_{eq} = Q/|\Delta V|$. Using the fact that the potentials add in series,

$$\frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2}$$

and so the equivalent capacitance for two capacitors in series becomes

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

(8)

The generalization to any number of capacitors connected in series is

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N} = \sum_{i=1}^{N} \frac{1}{C_i} \quad \text{(series)}$$

(9)
Problem 2: Capacitor Filled with Dielectric

a) A parallel-plate capacitor of area $A$ and spacing $d$ is filled with three dielectrics as shown in the figure below. Each occupies 1/3 of the volume. What is the capacitance of this system? [Hint: Consider an equivalent system to be three parallel capacitors, and justify this assumption.] Show that you obtain the proper limits as the dielectric constants approach unity, $\kappa_i \to 1$.]

Answer:
b) Suppose the capacitor is filled as shown in the figure below. What is its capacitance? Use Gauss's law to find the field in each dielectric, and then calculate $\Delta V$ across the entire capacitor. Again, check your answer as the dielectric constants approach unity, $\kappa_i \rightarrow 1$. Could you have assumed that this system is equivalent to three capacitors in series?

Answer: