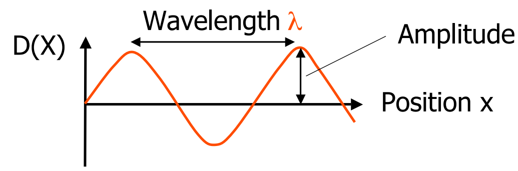
Waves

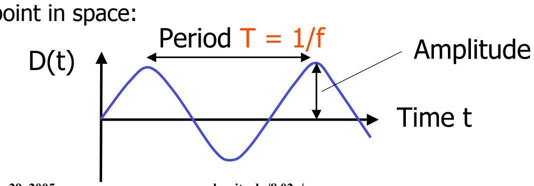
Roger Ramjet/www.surfcam.com

Reminder on waves

At a moment in time:



At a point in space:



Wave Equation

- Wave equation:

$$\frac{\partial^2 D(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 D(x,t)}{\partial t^2}$$

Couples variation in time and space

- Speed of propagation: $v = \lambda f$
- *How can we derive a wave equation from Maxwells equations?*

Wave properties

- What do we want to know about waves:
  - Speed of propagation?
  - Transverse or longitudinal oscillation?
  - What is oscillating?
  - What are typical frequencies/wavelengths?

Back to Maxwell's equation

- Wave equation is differential equation
- M.E. (so far) describe integrals of fields

$$\oint_{A_{closed}} \vec{E}\cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$
$$\oint_{L_{closed}} \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$
$$\oint_{A_{closed}} \vec{B}\cdot d\vec{A} = 0$$
$$\oint_{L_{closed}} \vec{B}\cdot d\vec{l} = \mu_0 I_{encl} + \mu_0\,\epsilon_0\frac{d\Phi_E}{dt}$$

Transform into differential equ's

Gauss Theorem

Flux/Unit Volume

$$\int_{V(A)} \overbrace{\vec{\nabla}\cdot\vec{F}}\,dV = \oint_A \vec{F}\cdot d\vec{A} = \Phi_F$$

Divergence → $\vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

Stokes Theorem

Loop Integral/Unit Area of Loop

$$\oint_L \vec{F} d\vec{l} = \int_{A(L)} \overbrace{\vec{\nabla} \times \vec{F}}\, d\vec{A}$$

Curl

$$\vec{\nabla} \times \vec{F} = \vec{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \vec{j}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \vec{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

Differential Form of M.E.

$$\oint_{A_{closed}} \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$
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$$\oint_{L_{closed}} \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

Gauss, Stokes

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

Differential Form of M.E.

Flux/Unit Volume

Charge Density

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
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Loop Integral/Unit Area

Current Density

Differential Form of M.E.

- Q: Do we need ρ and $\vec{j}$ to understand E.M. waves?

Differential Form of M.E.

- Q: Do we need ρ and $\vec{j}$ to understand E.M. waves?
- A: No! Light travels from sun to earth, i.e. in vacuum (no charge, no current)!
- There's no 'medium' involved!?
  - unlike waves on water or sound waves

Maxwell's Equations in Vacuum

- Look at Maxwell's Equations without charges, currents

$$\vec{\nabla} \cdot \vec{E} = 0$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

Now completely symmetric!

Maxwell's Equations in Vacuum

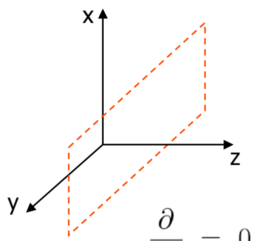
Solve for a simple geometry

$$\text{I.}\quad \vec{\nabla}\cdot\vec{E} = 0$$

$$\text{II.}\quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\text{III.}\quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{VI.}\quad \vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$



Allow variations only in z-direction:

$$\frac{\partial}{\partial x} = 0$$
$$\frac{\partial}{\partial y} = 0$$

Illustration

Roger Ramjet/www.surfcam.com

2-D wave:
x,z,D(x,z,t)

$$\frac{\partial}{\partial x} = 0$$

Maxwell's Equations in Vacuum

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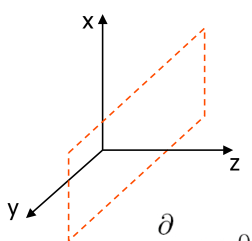

Allow variations only in z-direction:

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Electromagnetic Waves

- Note: (E_x, B_y) and (E_y, B_x) independent:

E_y, B_x

$$\frac{\partial B_x}{\partial z} = \frac{1}{c^2}\frac{\partial E_y}{\partial t}$$
$$\frac{\partial B_y}{\partial z} = -\frac{1}{c^2}\frac{\partial E_x}{\partial t}$$
$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$
$$\frac{\partial E_y}{\partial z} = \frac{\partial B_x}{\partial t}$$

E_x, B_y

Electromagnetic Waves

- We found wave equations:

$$\frac{\partial^2 B_y}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2}$$
$$\frac{\partial^2 E_y}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2}$$

same for E_x, B_x

v = c

E and B are oscillating!

Electromagnetic Waves

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E_x, B_y

$\vec{E} \perp \vec{B}$

Plane waves

- Example solution: Plane waves

$$
\begin{aligned}
E_y &= E_0\cos(kz-\omega t)\\
B_x &= B_0\cos(kz-\omega t)\\
\text{with } k &= \frac{2\pi}{\lambda}, \omega = 2\pi f \text{ and } f\lambda = c.
\end{aligned}
$$

- We can express other functions as linear combinations of sin,cos
  - 'White' light is combination of waves of different frequency
  - Demo...

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Check

$$
\begin{aligned}
\frac{\partial^2 B_y}{\partial z^2} &= \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2}\\
\frac{\partial^2 E_y}{\partial z^2} &= \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2}
\end{aligned}
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\frac{\partial E_x}{\partial z} &= -\frac{\partial B_y}{\partial t}\\
\frac{\partial E_y}{\partial z} &= \frac{\partial B_x}{\partial t}
\end{aligned}
$$

$$
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\frac{\partial^2 B_y}{\partial z^2} &= \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2}\\
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\end{aligned}
$$

$$
\begin{aligned}
&-kE_0\sin(kz-\omega t) = \omega B_0\sin(kz-\omega t)\\
\Rightarrow\ &\frac{|E_0|}{|B_0|} = \frac{k}{\omega} = c
\end{aligned}
$$

E.M. Wave Summary

- $\vec{E} \perp \vec{B}$ and perpendicular to direction of propagation
- Transverse waves
- Speed of propagation $v = c = \lambda f$
- $|\vec{E}|/|\vec{B}| = c$
- E.M. waves travel without medium