8.044, Spring 2006

More on Problem 4, PS3

The probablity density has been found to be

$$p\left(v_x\right) = \frac{4}{\sqrt{\pi}}\,\left(2\sigma^2\right)^{-3/2} v_x^2\;\exp\left[-\frac{v_x^2}{2\sigma^2}\right]$$

(some of us prefer the above form for reasons that might become clear later).

The needed improper definite integrals are of the form

$$\int_0^\infty u^2\,e^{-u^2}\,du = \frac{\sqrt{\pi}}{4},\qquad \int_0^\infty u^3\,e^{-u^2}\,du = \frac{1}{2},\qquad \int_0^\infty u^4\,e^{-u^2}\,du = \frac{3\sqrt{\pi}}{8};$$

these integrals have been treated extensively elsewhere and will not be rederived.

We have then, using basic calculus techinques to account for the needed factors of $2\sigma^2$,

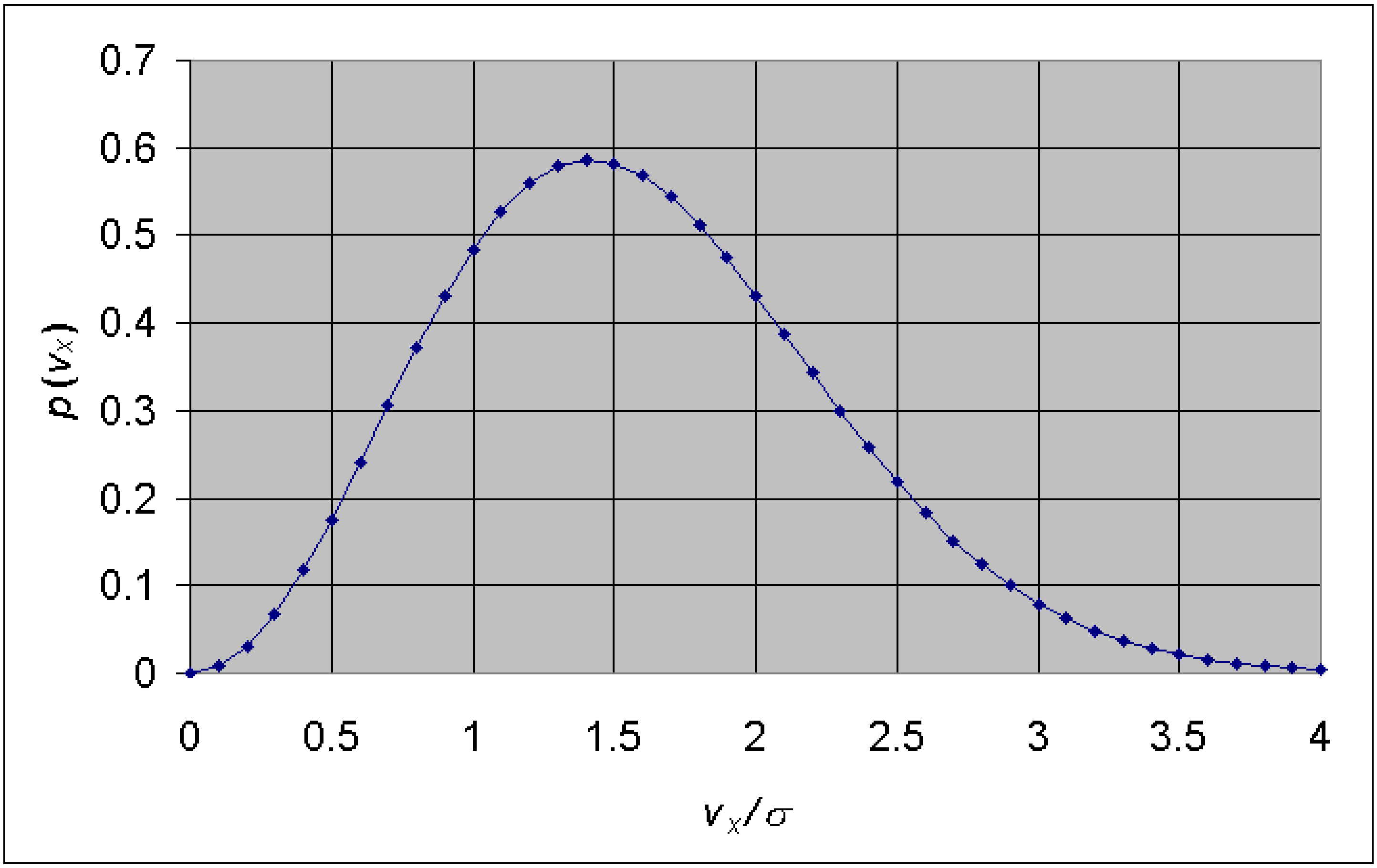
$$\begin{aligned}\langle v_x\rangle &= \int_0^\infty \frac{4}{\sqrt{\pi}}\,\left(2\sigma^2\right)^{-3/2} v_x^3\;\exp\left[-\frac{v_x^2}{2\sigma^2}\right]\,dv_x = \frac{4}{\sqrt{\pi}}\,\left(2\sigma^2\right)^{-3/2}\,\left(2\sigma^2\right)^2\,\frac{1}{2}\\ &= \frac{2}{\sqrt{\pi}}\left(\sqrt{2}\sigma\right)\\ \langle v_x^2\rangle &= \int_0^\infty \frac{4}{\sqrt{\pi}}\,\left(2\sigma^2\right)^{-3/2} v_x^4\;\exp\left[-\frac{v_x^2}{2\sigma^2}\right]\,dv_x = \frac{4}{\sqrt{\pi}}\,\left(2\sigma^2\right)^{-3/2}\,\left(2\sigma^2\right)^{5/2}\,\frac{3\sqrt{\pi}}{8}\\ &= 3\,\sigma^2.\end{aligned}$$

From this, we see that

$$\mathrm{Var}\left(v_x\right) = \left(3 - \frac{8}{\pi}\right)\sigma^2 = 0.4535\,\sigma^2,\qquad \text{stand. dev.}\left(v_x\right) = 0.6734\,\sigma^2;$$

in this case, the parameter σ is *not* the standard deviation of the probability distrtibution.

A plot of $p\left(v_x\right)$ as a function of v_x is on the following page.



What can clearly be seen is the maximum of $p\,(v_x)$ is attained at the *most likely speed*, easily calculated as $\sqrt{2}\sigma$, slightly less than the mean $\langle v_x \rangle = \sqrt{8/\pi}\,\sigma \sim 1.596\,\sigma$. It is also seen that the range from the mean $-/+$ the standard deviation, from $\sim 0.92\sigma$ to $\sim 2.27\sigma$ is a reasonable estimate of the "spread" of the probability distribution.

The above plot was generated using Excel, the default spreadsheet program for Windows. Use of a spreadsheet allows numerical calculation of the mean and standard deviation. Excel is maybe not the best program for plotting, but it's fairly common.