A Possibly Too-Slick Way to Find the Full Width at Half-Maximum of a Driven Oscillator

For concreteness, assume a driven LRC circuit with the usual notation. If the driving voltage is $V(t) = V_0 \cos \omega t$, the time-averaged power is

$$\langle P \rangle_t = \frac{1}{2} \frac{R\,V_0^2}{R^2 + \frac{L}{C}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2} = \frac{1}{2} \frac{R\,V_0^2}{Z^2(\omega)},$$

where $\omega_0 \equiv 1/\sqrt{LC}$. The above form for $Z^2(\omega)$ is not the most common one, but it serves our purposes best. First, notice that the ratio L/C has the same dimensions as impedance squared, so we'll give it a temporary name, $\frac{L}{C} \equiv Z_0^2$; temporary because Z_0 is often reserved for the "impedance of free space," $\sqrt{\frac{\mu_0}{\epsilon_0}} = 377\,\Omega$.

So, we have

$$Z^2(\omega) = R^2 + Z_0^2 \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2$$

from which we see immediately that $Z^2(\omega)$ is minimized, and hence the power maximized, at $\omega = \omega_0$, the well-known result for driven oscillators.

The FWHM is $\omega_+ - \omega_-$ where

$$Z^2(\omega_+) = Z^2(\omega_-) = 2\,R^2, \qquad \omega_+ > \omega_-,$$

so that ω_+ and ω_- are the *positive* solutions to

$$\left| \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right| = \frac{R}{Z_0} = \frac{1}{Q},$$

where the quality factor Q is introduced to show that this calculation of the FWHM can be extended to more general oscillators.

Here's where the above form for $Z^2(\omega)$ pays off; note that

$$Z^2(\omega) = Z^2\left(\frac{\omega_0^2}{\omega}\right),$$

from which we infer immediately that

$$\omega_- = \frac{\omega_0^2}{\omega_+},$$

$$\begin{aligned}
\text{FWHM} &= \omega_+ - \omega_- \\
&= \omega_+ - \frac{\omega_0^2}{\omega_+} \\
&= \omega_0 \left(\frac{\omega_+}{\omega_0} - \frac{\omega_0}{\omega_+} \right) \\
&= \frac{\omega_0}{Q}
\end{aligned}$$

and we're done.

As long as we're being slick, re-express Z^2 as

$$Z^2(\omega) = R^2 + 4\, Z_0^2\, \sinh^2 \eta, \qquad \eta \equiv \ln \left(\frac{\omega}{\omega_0} \right).$$

The condition for resonance is then

$$\sinh^2 \eta = \frac{1}{4\, Q^2},$$

which has two roots given by

$$\sinh \eta_+ = \frac{1}{2\, Q}, \qquad \sinh \eta_- = -\frac{1}{2\, Q}$$

from which $\eta_+ = -\eta_-$. Thus armed,

$$\begin{aligned}
\text{FWHM} &= \omega_+ - \omega_- \\
&= \omega_0 \left(e^{\eta_+} - e^{\eta_-} \right) \\
&= \omega_0 \left(e^{\eta_+} - e^{-\eta_+} \right) \\
&= \omega_0\, 2 \sinh \eta_+ \\
&= \frac{\omega_0}{Q}
\end{aligned}$$

and we are done yet again.

Of course, this is the same as the algebraic way, just hiding the algebra in the properties of the hyperbolic function. The justification is that Z^2 is a symmetric funtion of η but not of ω.