Using a Sommerfeld-Watson Transformation

to Investigate a Certain Aspect of a

Solution of Laplace's Equation

Disclaimer: These notes are a bit outside of **8.03 at MIT**, and might even be considered a bit of a novelty. The details of the Sommerfeld-Watson transformation will not be given; if anything, an easier form of the transformation could be used to find the sums given in the notes **Some Sums**. However, these notes show the kind of trick that we have up our collective sleeve.

Consider the boundary-value Laplace's Equation for $T(x, y)$ on the square $0 < x < 1$, $0 < y < 1$,

$$\nabla^2 T = 0, \qquad T(x, 0) = 1, \qquad T(1, y) = T(x, 1) = T(0, y) = 0.$$

Note that the function T, easily taken as "temperature," has been scaled to unity at $y = 0$, as have the dimensions of the square.

The formal solution is presented below, with admittedly many skipped steps. Specifically, if we look for an *orthogonal function expansion* of the form

$$T(x, y) = \sum_n \sin n\pi x \left(A_n\, e^{n\pi y} + B_n\, e^{-n\pi y}\right),$$

and apply the boundary conditions, we find that

$$1 = \sum_n (A_n + B_n) \sin n\pi x,$$
$$0 = \sum_n \left(A_n\, e^{n\pi} + B_n\, e^{-n\pi}\right) \sin n\pi x.$$

Note that the zero boundary conditions at $x = 0$, $x = 1$ are already satisfied, in that only $\sin n\pi x$, not $\cos n\pi x$, are included in the sum; this is indeed one of the skipped steps. In fact, right now we'll refer to B&B's Equation (2.80) (but make oh-so-sure to note that in Figure (2.2), positive is down. Oops.) to cite the result

$$A_n + B_n = \frac{4}{n\pi}, \quad n \text{ odd}; \qquad A_n + B_n = 0, \quad n \text{ even}.$$

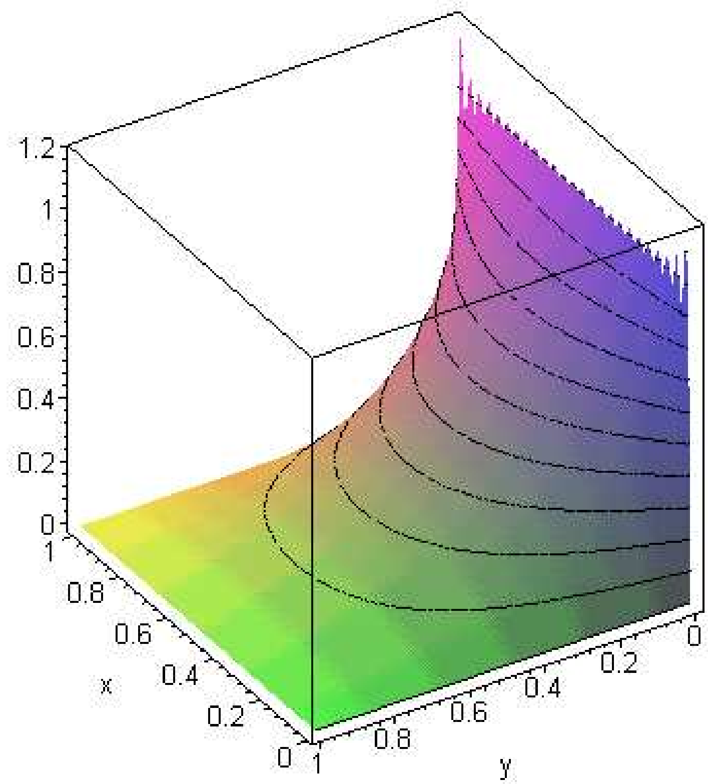
The boundary condition at $y = 1$ gives each term in the sum identically zero, or

$$A_n\, e^{n\pi} + B_n\, e^{-n\pi} = 0, \quad \text{all } n.$$

It is now possible to solve for the formal solution. As promised, several steps are about to be skipped, but the result may be expressed as

$$T(x,\, y) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n \sinh n\pi} \sin n\pi x \, \sinh\left(n\pi(1-y)\right);$$

note that because $A_n = B_n = 0$ identically for n even, the sum is over n odd only. A contour plot of the first 50 terms is shown below. The *Gibbs phenomenon* is clearly seen at $(x = 0,\, y = 0)$ and $(x = 1,\, y = 0)$.



At this point, both from the figure and by invoking some hindsight, if we had solved the same problem with the boundary conditions

$$T\left(0,\, 0\right), \qquad T\left(0,\, 1\right) = 1,$$

which is completely equivalent, with $y \to 1-y$, we could then seek a formal solution of the form

$$T(x,\, y) = \sum_{n} C_n \, \sin n\pi x \, \sinh n\pi y,$$

resulting in

$$C_n \sinh n\pi y = \frac{1}{4\pi\, n}, \qquad n \text{ odd}; \qquad C_n = 0, \qquad n \text{ even},$$

leading of course to the same result.

So, where does a Sommerfeld-Watson transformation come in? Often, when determination of the above formal series solution is assigned as a problem, an auxiliary part is to "Find $T(1/2,\,1/2)$." Okay, let's try. It will help to use the double-argument expression for hyperbolic sine,

$$\sinh n\pi = 2\,\sinh n\pi/2\,\cosh n\pi/2$$

(remember where you saw it first) to find, formally,

$$T\left(\frac{1}{2},\,\frac{1}{2}\right) = \frac{2}{\pi}\sum_{n\text{ odd}}\frac{1}{n}\frac{\sin n\pi/2}{\cosh n\pi/2}.$$

To evaluate this beast, consider the function

$$f(z) = \frac{1}{z}\frac{1}{\cos z\pi/2}\frac{1}{\cosh z\pi/2}$$

and evaluate the *residues* of $f(z)$. Without going into too much detail (this is complex analysis), a non-zero residue is obtained at any point a in the complex plane where $f(z)$ is singular, and is equal to the (non-zero) coefficient of $1/(z-a)$ in the *Laurent series expansion of $f(z)$ about a* (you may want to look this up) if the series exists (not all do).

Anyway, our function $f(z)$ is singular at the origin (a gimme), all odd integers (positive *and* negative) and, here's the kicker, at all odd integer multiples of i; recall $\cos(i\eta) = \cosh\eta$ and $\cosh(i\eta) = \cos\eta$ (remember, we use $\sqrt{-1} = i$ in this class). A calculation that is not simple, but far from prohibitively difficult, shows that for each positive integer, the residue is exactly the negative of the corresponding term we need, and yes, the residue vanishes at positive even integers ($f(z)$ is not singular at these points). The residues at the corresponding negative values are the same. By the way, the factor of $\pi/2$ in $1/(\cos z\pi/2)$ pops into the residues, as seen by taking the Laurent Series, and takes care of the $2/\pi$ in the sum we want (see below). We're that good.

The great thing is the residues at the integral multiples of i give exactly the same result. So, we take the sum of all of these residues and divide by 4 and set this equal to the negative of the residue at the origin. Why we can do this is, again, a result of complex analysis, although it's strongly related to Green's Theorem in the plane. The residue at the origin is the easiest to calculate, and is equal to just plain 1. Thus, our sum is equal to a very simple $1/4$, one of our favorite rational numbers.

So, that's neat, but what's the very large deal? Well, if we just wanted to give the answer to the temperature in the center of the plate, we would only need to invoke symmetry. That is, redo the problem where $T = 1$ everywhere on the boundary of the square (trivially, $T = 1$ everywhere in the interior for this boundary), argue that the temperature in the center for our problem therefore has to be a quarter of this, and we're done.

If this all sounds too hifalutin', recall that in the notes **Some Sums**, we did the same sort of thing, in that we found Fourier Series for functions that could be evaluated easily at special points, equated the values to the corresponding sums, and then declared that we had found the desired sums. Well, all of those sums could be found by Sommerfeld-Watson transformations as well, but we let it go. You make the call.

The Residue We've Always Wanted

For integer n, let $\delta = z - n\pi/2$. The introduced variable δ need not be small, but no other residues of $f(z)$ should be within $|\delta|$ of n. Simple trig gives

$$\cos\frac{z\pi}{2} = \cos\left[\frac{n\pi}{2} + \frac{\delta\pi}{2}\right] = -\sin\frac{n\pi}{2}\sin\frac{\delta\pi}{2}.$$

The reciprocal of this expression appears in $f(z)$, and another fairly simple calculation gives

$$\frac{1}{\sin\delta\pi/2} = \frac{2}{\delta\pi} + T(\delta),$$

where $T(\delta)$ is a Taylor Series in δ (not to be confused withe original $T(x,\, y)$). Thus, the residue of $f(z)$ at $z = n$ is exactly the term of the formal sum that we need.