Blackbody Radiation

As derived on Page 312 of B&B, the energy density of electromagnetic waves in thermal equilibrium with their surroundings is given by

$$U(T) = a\,T^4,$$

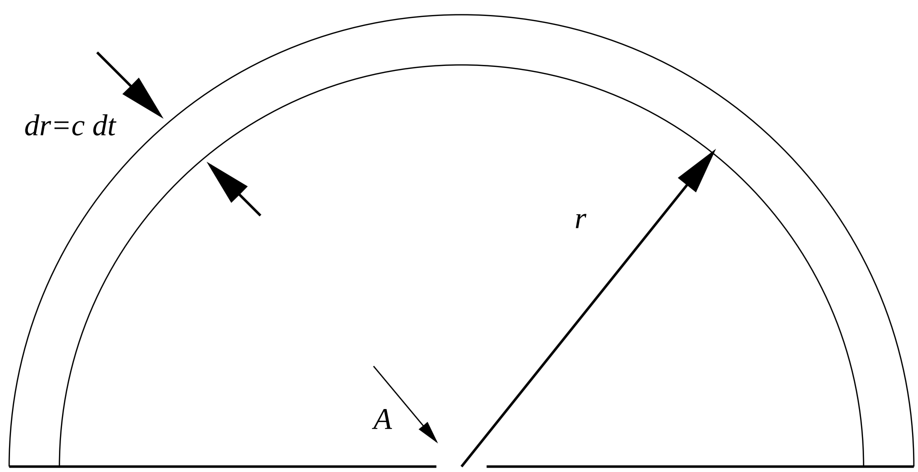
where

$$a = \frac{8\pi^5 k^4}{15 h^3 c^3} = \frac{\pi^2 k^4}{15 \hbar^3 c^3}$$

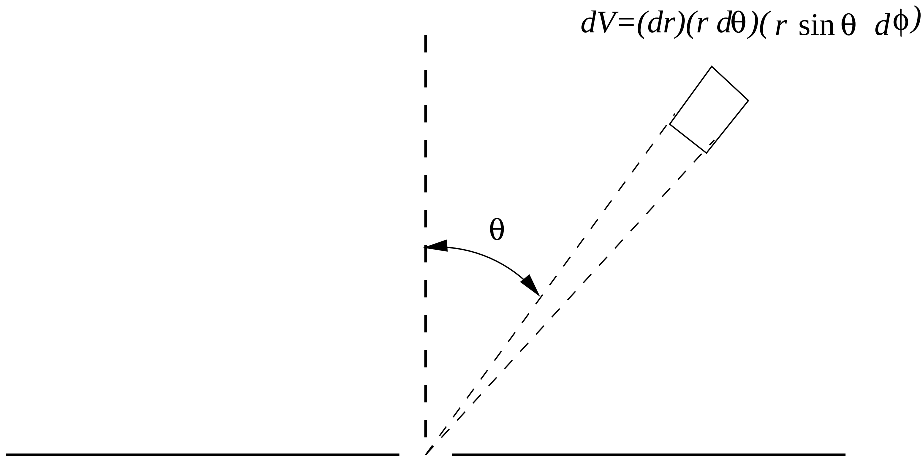
and the substitution $\hbar \equiv h/2\pi$ has been made. These waves will be a source of radiation. To find the intensity of this radiation (power per unit area), consider a boundary between two regions, one at temperature T_1 and the other at T_2. If a small hole is made in this boundary, energy will flow from the higher temperature region to the lower (if $T_1 = T_2$, the regions are in equilibrium and there is no net energy flow either way). In this case, "small" means that the presence of the opening does not affect either T_1 or T_2, and that the radiation from one region to the other depends only on the temperature of the region where the radiation originates.

So, we will consider a region of temperature T separated from a vacuum by a boundary, with a small hole of area A. We will need to make two further assumptions; first, that the energy is associated with electromagnetic waves in the absence of charges, and hence is propagated at the speed of light c, and second that the radiation is *isotropic* (*iso* = same, *trop* = direction), meaning that the radiation intensity is the same in all direcitons.

So, then, any radiation that leaves the opening at a given time was at a distance $r = ct$ a time t previously. This set of pionts is a hemisphere, shown in cross-section.



The energy leaving the opening in time dt must have come from a shell of thickness $c\,dt$. However, not all of this energy in the shell will leave through the opening. Using the isotropic property of the radiation, consider a volume dV, the usual volume element in spherical polar coordinates with origin at the opening and azimuth (z-axis) perpendicular to the boundary.



The electromagnetic energy in this volume is $dE = U\,dV$ (here, "E" is energy, not electric field amplitude). This radiation will radiate isotropically; the fraction that leaves through the opening will be the ratio of the solid angle subtended by the opening *as seen from the point where the volume element dV is* to the total solid angle. This ratio is $A\cos\theta/4\pi r^2$. Thus,

$$\frac{A\cos\theta}{4\pi r^2}\,dE = A\,dS\,dt, \quad \text{or} \quad \frac{\cos\theta}{4\pi r^2}\,dV\,U = dS\,dt.$$

Using

$$dV = r^2\,\sin\theta\,dr\,d\theta\,d\phi = r^2\,\sin\theta\,d\theta\,d\phi\,c\,dt,$$

$$dS = \frac{Uc}{4\pi}\cos\theta\sin\theta\,d\theta\,d\phi \quad \text{and}$$

$$\begin{aligned} S = \int dS &= \frac{Uc}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi/2}\sin\theta\cos\theta\,d\theta \\ &= \frac{Uc}{2}\int_0^{\pi/2}\sin\theta\,d(\sin\theta) \\ &= \frac{Uc}{2}\frac{1}{2} = \frac{Uc}{4}. \end{aligned}$$

(Note the range of integration on θ, corresponding to half a sphere). Thus,

$$S = \frac{Uc}{4} = \frac{c}{4}\,a\,T^4 = \sigma\,T^4, \quad \text{where}$$

$$\sigma = \frac{c}{4}\, a = \frac{\pi^2\, k^4}{60\hbar^3 c^2} = 5.67051 \times 10^{-8}\ \frac{\mathrm{W}}{\mathrm{m}^2 \cdot \mathrm{K}^4}$$

is the "Stefan-Boltzmann" constant.

Thus, the *net* rate of heat flow across an aperture of area A between two regions of radiation at temperatures T_1 and T_2 is

$$H = \sigma\, A\, \left(T_1^4 - T_2^4\right).$$

Note; You need the result

$$\frac{S_1}{S_2} = \left(\frac{T_1}{T_2}\right)^4$$

to do B&B problem 4.6. The numbers in that problem have been rigged! Calculators are not necessary!