Adiabatic Bulk Moduli

To find the speed of sound in a gas, or any property of a gas involving elasticity (see the discussion of the Helmholtz oscillator, B&B page 22, or B&B problem 1.6, or French Pages 57-59.), we need the "bulk modulus" of the fluid. This will correspond to the "spring constant" of a spring, and will give the magnitude of the restoring agency (pressure for a gas, force for a spring) in terms of the change in physical dimension (volume for a gas, length for a spring). It turns out to be more useful to use an *intensive* quantity for the bulk modulus of a gas, so what we want is the change in pressure per fractional change in volume, so the bulk modulus, denoted as κ (the Greek "kappa" which, when written, has a great tendency to look like k, and in fact French uses "K"), is

$$\kappa = -\frac{\Delta p}{\Delta V / V}, \quad \text{or} \quad \kappa = -V \frac{dp}{dV}.$$

The minus sign indicates that for normal fluids (not all are!), a negative change in volume results in an increase in pressure.

To find the bulk modulus, we need to know something about the gas and how it behaves. For our purposes, we will need three basic principles (actually 2 1/2) which we get from thermodynamics, or the kinetic theory of gasses. You might well have encountered these in previous classes, such as chemistry.

A) The ideal gas law; $pV = nRT$, with the standard terminology.

B) The first law of thermodynamics; $dU = dQ - p\,dV$, where U is internal energy and Q is heat.

C) (a consequence of B, so call it B 1/2) The internal energy of a gas is a state variable, and hence is a function of temperature. This allows us to say that $dU = n\,C_V\,dT$, where C_V is the molar heat capacity at constant volume. What this means is that any process that changes U will change U by the same amount as a constant-volume process with the same temperature change would. Specifically, for a constant-pressure process, $dU = n\,C_p\,dT - p\,dV$, where C_p is the molar heat capacity at constant pressure.

So, we have two expressions for dU, and they must be the same;

$$n\,C_V\,dT = n\,C_p\,dT - p\,dV.$$

Using the ideal gas law for a constant-pressure process $(dp = 0)$, $n\,R\,dT = p\,dV$, so the above relation between C_V and C_p becomes

$$n\,C_V\,dT = n\,C_p\,dT - n\,R\,dT, \quad \text{or} \quad C_p = C_V + R.$$

From this, we se that $C_p > C_V$. This is what is expected from physical grounds; if a gas is free to expand (constant pressure), it takes more heat to increase the temperature than if the gas is confined. It will be convenient to define the quantity $\gamma \equiv \frac{C_p}{C_V} > 1$.

For an adiabatic process, no heat is added, and $dQ = 0$. Note that this does *not* mean that $dT = 0$; in fact, for an adiabatic process, none of the intensive quantities p, T or the molar volume V/n will be constant. Equating expressions for dU,

$$dU = n\,C_V\,dT = -p\,dV;$$

from $pV = n\,RT$,

$$n\,R\,dT = (V\,dp + p\,dV),$$

so upon substitution of dT, rearrangement, and cancellation of the factor n,

$$C_V\,(V\,dp + p\,dV) = -R\,p\,dV$$
$$C_V\,V\,dp = -(R + C_V)\,p\,dV$$
$$V\,dp = -\gamma\,p\,dV.$$

At this point, we can get our result for the adiabatic bulk modulus,

$$\kappa = -V\frac{dp}{dV} = \gamma\,p.$$

We can also get a more general result relating the pressure and volume for an adiabatic process,

$$p\,V^\gamma = \text{constant},$$

which is useful when a calculation of work, the integral of $p\,dV$, is needed.

In practice, isothermal $(dT = 0)$ processes for gasses are rather rare; adding heat to a system in such a way that the temperature remains uniform is difficult, and is often just an idealization. For such a process, however, the isothermal bulk modulus is simply $\kappa = p$. Note that this corresponds to $\gamma = 1$, or a system in which the heat capacities at constant volume and constant pressure are nearly identical. This is often the case for condensed matter, but not for gasses.