Coupled Linear Oscillators

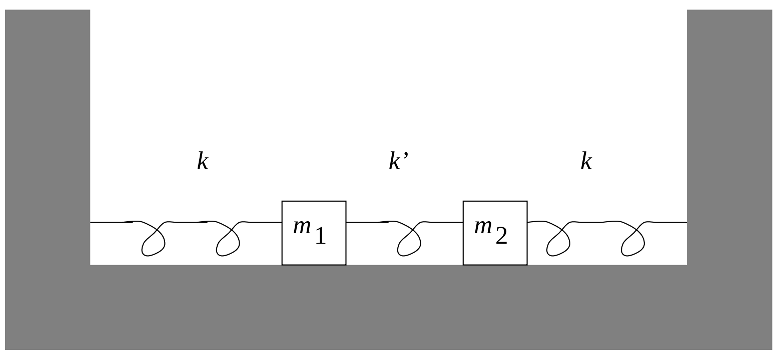
You need not memorize any of this, but please don't toss it; we'll be needing some of the results soon.

In these notes, matrices will be denoted by boldface captial letters.

For a system of coupled oscillators subject to linear restoring forces, we found that the equations of motion many be written in matrix form;

$$\mathbf{M}\,\ddot{\vec{x}}(t) = \mathbf{K}\,\vec{x}(t), \tag{1}$$

where $\vec{x}(t)$ will be taken as a column vector, each of whose components is one of the dependent variables, $\mathbf{M}$ is a diagonal matrix with positive elements, and $\mathbf{K}$ is symmetric. For example, we have seen that for the simple system of two masses connected by springs, as shown,



the coupled equations

$$m_1\ddot{x}_1 = -kx_1 - k'(x_1 - x_2)$$
$$m_2\ddot{x}_2 = -kx_2 - k'(x_2 - x_1)$$

became

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -k-k' & k' \\ k' & -k-k' \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

It will be helpful to use row vectors also; if

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad , \quad \vec{x}^T = [x_1, \ldots, x_n]\,.$$

The "T" superscript is for "transpose". From the symmetry of $\mathbf{M}$ and $\mathbf{K}$, note that

$$\vec{x}^T\mathbf{M} = (\mathbf{M}\,\vec{x})^T \quad , \quad \vec{x}^T\mathbf{K} = (\mathbf{K}\,\vec{x})^T\,.$$

Note also that $\vec{x}^T\vec{x}$ is a *scalar*; in fact, $\vec{x}^T\vec{x} = \vec{x}\cdot\vec{x}$.

So, let's look for solutions to (1) of the form

$$\ddot{\vec{x}} = -\omega^2\vec{x}. \tag{2}$$

Note that, if $\vec{x}_0$ is a constant vector, $\vec{x} = \vec{x}_0 e^{i\omega t}$, $\vec{x} = \vec{x}_0 e^{-i\omega t}$, $\vec{x} = \vec{x}_0\cos\omega t$, $\vec{x} = \vec{x}_0\sin\omega t$ or any linear combination will suffice. Note that in this usage, $\vec{x}_0$ is not necessarily $\vec{x}(0)$. The specific functional form won't matter, as long as (2) holds.

Substitution into (1) yields

$$\left(\mathbf{K} + \omega^2\mathbf{M}\right)\vec{x} = \vec{0}. \tag{3}$$

The only way for non-trivial solutions of (3) to exist is to have $\det\left(\mathbf{K} + \omega^2\mathbf{M}\right) = 0$. This allows us to solve algebraically for ω^2, and hence for $\vec{x}_0$ (within constant multiples). Specifically, let $\vec{x}_\alpha = \vec{x}_{\alpha 0}e^{-i\omega_\alpha t}$. Now, then, what are meant by "normal modes"? Who are we to judge anyone else's normalcy? This is, after all MIT. Well, consider

$$\vec{x}^T_{\beta 0}\,\mathbf{K}\,\vec{x}_{\alpha 0} = \vec{x}^T_{\beta 0}\left(\mathbf{K}\,\vec{x}_{\alpha 0}\right) = \vec{x}^T_{\beta 0}\left(-\omega^2_\alpha\,\mathbf{M}\,\vec{x}_{\alpha 0}\right) = -\omega^2_\alpha\left(\vec{x}^T_{\beta 0}\,\mathbf{M}\,\vec{x}_{\alpha 0}\right).$$

But,

$$\begin{aligned}\vec{x}^T_{\beta 0}\,\mathbf{K}\,\vec{x}_{\alpha 0} &= \left(\vec{x}^T_{\beta 0}\,\mathbf{K}\right)\vec{x}_{\alpha 0} = \left(\mathbf{K}\,\vec{x}_{\beta 0}\right)^T\vec{x}_{\alpha 0}\\ &= \left(-\omega^2_\beta\,\mathbf{M}\,\vec{x}_{\beta 0}\right)^T\vec{x}_{\alpha 0} = -\omega^2_\beta\left(\mathbf{M}\,\vec{x}_{\beta 0}\right)^T\vec{x}_{\alpha 0}\\ &= -\omega^2_\beta\left(\vec{x}^T_{\beta 0}\,\mathbf{M}\,\vec{x}_{\alpha 0}\right).\end{aligned}$$

So, we have

$$-\omega^2_\alpha\left(\vec{x}^T_{\beta 0}\,\mathbf{M}\,\vec{x}_{\alpha 0}\right) = -\omega^2_\beta\left(\vec{x}^T_{\beta 0}\,\mathbf{M}\,\vec{x}_{\alpha 0}\right), \quad \text{or}$$

$$\left(\omega^2_\alpha - \omega^2_\beta\right)\left(\vec{x}^T_{\beta 0}\,\mathbf{M}\,\vec{x}_{\alpha 0}\right) = 0,$$

which means that if $\omega^2_\alpha \neq \omega^2_\beta$,

$$\vec{x}^T_{\beta 0}\,\mathbf{M}\,\vec{x}_{\alpha 0} = 0,$$

and this is the interpretation of "normal modes" that we need.

If $\omega^2_\alpha = \omega^2_\beta$ for some $\alpha \neq \beta$, the above relation still holds, if $\vec{x}_{\alpha 0}$ and $\vec{x}_{\beta 0}$ are chosen properly. First, note that if $\omega^2_\alpha = \omega^2_\beta$, (3) is satisfied for $\omega^2 = \omega^2_\alpha$ and $\vec{x} = a\vec{x}_{\alpha 0} + b\vec{x}_{\beta 0}$ for any scalars a and b. Thus, $\vec{x}_{\alpha 0}$ and $\vec{x}_{\beta 0}$ may be chosen such that $\vec{x}^T_{\beta 0}\,\mathbf{M}\,\vec{x}_{\alpha 0} = 0$ if $\vec{x}_{\alpha 0}$ and $\vec{x}_{\beta 0}$ are linearly independent, and a result from linear algebra shows that they will be. However, the physics shows why the vectors $\vec{x}_{\alpha 0}$ and $\vec{x}_{\beta 0}$ may *always* be chosen to be normal.

The specific values of ω_α^2 and the components of $\vec{x}_{\alpha 0}$ will depend on the values of the masses (*i.e.*, the diagonal elements of $\mathbf{M}$). If any frequency appears as a multiple root of (3), changing a value of some mass *slightly* will remove the degeneracy, the ω_α^2 will be different, and the modes will be necessarily normal. A small change in the elements of $\mathbf{M}$ cannot change the normality of the vectors. Please note that this explanation does not constitute a proof; it relies on our belief, on physical grounds, that a small change in the linear system cannot grossly affect the normalcy criteria.

We saw in class that if the masses are all the same, $\mathbf{M}$ is a scalar multiple of the indentity matrix, and the normality condition reduces to $\vec{x}_{\beta 0}\cdot\vec{x}_{\alpha 0} = 0$. We also saw that if the masses are *not* identical, the modes are normal when "weighted" by the masses.

Before moving on, let's take advantage of the physics to help us do some math; define the matrix $\mathbf{M}^{\frac{1}{2}}$ as the diagonal matrix whose elements are the non-negative square roots of the masses, so that $\mathbf{M}^{\frac{1}{2}}\,\mathbf{M}^{\frac{1}{2}} = \mathbf{M}$. Then, rewrite (1) as

$$\mathbf{M}\,\ddot{\vec{x}} = \mathbf{M}^{\frac{1}{2}}\,\mathbf{M}^{\frac{1}{2}}\,\ddot{\vec{x}} = \mathbf{K}\,\vec{x} = \mathbf{K}\,\mathbf{M}^{-\frac{1}{2}}\,\mathbf{M}^{\frac{1}{2}}\,\vec{x} \quad , \quad \text{or}$$

$$\mathbf{M}^{\frac{1}{2}}\,\ddot{\vec{x}} = \left(\mathbf{M}^{-\frac{1}{2}}\,\mathbf{K}\,\mathbf{M}^{-\frac{1}{2}}\right)\mathbf{M}^{\frac{1}{2}}\,\vec{x},$$

where $\mathbf{M}^{-\frac{1}{2}} = \left(\mathbf{M}^{\frac{1}{2}}\right)^{-1}$. This last point may seem trivial, but fractional powers of matrices are *not* as easily defined or determined as for scalars; our form for $\mathbf{M}$ makes it easy. Also, note that if the masses are not the same, $\mathbf{M}^{-\frac{1}{2}}\,\mathbf{K} \neq \mathbf{K}\,\mathbf{M}^{-\frac{1}{2}}$. So, define new coordinates by $\vec{y}(t) = \mathbf{M}^{\frac{1}{2}}\,\vec{x}(t)$. With $\mathbf{K}' \equiv \mathbf{M}^{-\frac{1}{2}}\,\mathbf{K}\,\mathbf{M}^{-\frac{1}{2}}$, (1) becomes

$$\ddot{\vec{y}} = \mathbf{K}'\,\vec{y}. \tag{4}$$

Our conditions for normal modes are then

$$\det\left(\mathbf{K}' + \omega^2\mathbf{I}\right) = 0, \quad \vec{y}_{\alpha 0}\cdot\vec{y}_{\beta 0} = 0.$$

Note that this is a mathematical convenience; if the $x_1, \ldots, x_n$ represent lengths, changing to $y_1, \ldots, y_n$ changes each length by a *different* factor, and that ain't physics. We will want the mathematical convenience at a later date.

For our purposes, we will need real solutions, and it would be nice to know that real solutions exist; this would mean that ω_α is real for all α (if ω_α is real, all of the elements of $\vec{x}_{\alpha 0}$ may be chosen real). All we need is the fact that both $\mathbf{K}$ and $\mathbf{M}$ are real and symmetric. Then, if $\mathbf{K}\,\vec{x}_{\alpha 0} = -\omega_\alpha^2\,\mathbf{M}\,\vec{x}_{\alpha 0}$,

$$\vec{x}_{\alpha 0}^{T}\,\mathbf{K} = \left(\mathbf{K}\,\vec{x}_{\alpha 0}\right)^T = -\omega_\alpha^2\left(\mathbf{M}\,\vec{x}_{\alpha 0}\right)^T$$

and,

$$\left(\vec{x}_{\alpha 0}^{*T}\,\mathbf{K}\right) = \left(\mathbf{K}\,\vec{x}_{\alpha 0}^{*}\right)^T = \left(-\omega_\alpha^2 \mathbf{M}\,\vec{x}_\alpha\right)^{*T} = \left(-\omega_\alpha^2\right)^*\left(\vec{x}_{\alpha 0}^{*T}\,\mathbf{M}\right).$$

Thus, we have

$$\begin{aligned}\vec{x}_{\alpha 0}^{*T}\left(\mathbf{K}\vec{x}_{\alpha 0}\right) = \vec{x}_{\alpha 0}^{*T}\left(-\omega_\alpha^2\,\mathbf{M}\,\vec{x}_{\alpha 0}\right) &= -\omega_\alpha^2\left(\vec{x}_{\alpha 0}^{*T}\,\mathbf{M}\,\vec{x}_{\alpha 0}\right),\\ \left(\vec{x}_{\alpha 0}^{*T}\,\mathbf{K}\,\right)\vec{x}_{\alpha 0} = \left(-\omega_\alpha^2\right)^*\,\left(\vec{x}_{\alpha 0}^{*T}\,\mathbf{M}\right)\vec{x}_{\alpha 0} &= \left(-\omega_\alpha^2\right)^*\left(\vec{x}_{\alpha 0}^{*T}\,\mathbf{M}\,\vec{x}_{\alpha 0}\right),\end{aligned}$$

so $\left(\omega_\alpha^2\right)^* = \omega_\alpha^2$, and ω_α^2 is real. Those familiar with the terminology of linear algebra will note that we have introduced the Hermitian conjugates of the $\vec{x}_{\alpha 0}$, and that because both $\mathbf{K}$ and $\mathbf{M}$ are real and symmetric, they are Hermitian, as is $\mathbf{K}'$.

Showing that ω_α^2 is non-negative, so that ω_α is real, is best done by appeal to the phyiscs. Specifically, $\vec{x} = \vec{0}$ must be a *stable* configuration point (or neutrally stable, if $\omega_\alpha = 0$ for some α). A way to see this is to consider the potential energy $V(x_1, \dots, x_n)$ of the system due to the forces represented by $\mathbf{K}$; each component of $\mathbf{K}$ is then

$$\mathbf{K}_{jk} = -\frac{\partial^2 V}{\partial x_j \partial x_k}\bigg|_{\vec{x}=\vec{0}}.$$

Apart from the minus sign, $\mathbf{K}$ is a *Hessian* matrix, a term you might have encountered in 18.02 or the equivalent. We'll just use the calculus result, that V is a minimum at $\vec{x} = \vec{0}$ if the eigenvalues of $\mathbf{K}$ are negative. We could then use the result from linear algebra that if the eigenvalues of $\mathbf{K}$ are negative, the eigenvalues of $\mathbf{K}'$ (which are *not* the eigenvalues of $\mathbf{K}$) are negative. Or, we can define $V'(y_1, \dots, y_n) = V(\sqrt{m_1}y_1, \dots, \sqrt{m_n}y_n)$, using $y_i = \frac{x_1}{\sqrt{m_1}}$, as in (4), and then observe that if V is a minimum at $\vec{x} = \vec{0}$, V' is a minimum at $\vec{y} = \vec{0}$.

So, now that we know that they exist, let's look for *real* solutions (*e.g.*, sines and cosines), specifically

$$\vec{x}(t) = \sum_{j=1}^{n}\left(a_j \vec{x}_{j0}\cos\omega_j t + b_j \vec{x}_{j0}\sin\omega_j t\right), \quad \vec{x}(0) = \vec{x}_0, \quad \dot{\vec{x}}(0) = \vec{v}_0, \tag{5}$$

where the a_j, b_j are constants, and $\vec{x}_{j0}$ and ω_j are as found previously. Note that $\vec{x}_0$, without a j or α subscript, is the initial value $\vec{x}(0)$, and is *not*, in general, equal to $\vec{x}_{j0}$ for any j. In terms of the vectors $\vec{x}_{j0}$,

$$\vec{x}_0 = \sum_{j=1}^{n} a_j \vec{x}_{j0}, \quad \vec{v}_0 = \sum_{j=1}^{n}\omega_j b_j \vec{x}_{j0}.$$

To find the a_j and b_j, use the linearity of $\mathbf{M}$ and the normalcy condition;

$$\mathbf{M}\,\vec{x}_0 = \sum_{j=1}^{n} a_j\,\mathbf{M}\,\vec{x}_{j0}, \quad \vec{x}_{k0}^{T}\,\mathbf{M}\,\vec{x}_0 = \sum_{j=1}^{n} a_j \vec{x}_{k0}^{T}\,\mathbf{M}\,\vec{x}_{j0}.$$

The sum is of n terms, all but one of which vanish. That term will be for $j = k$, so

$$\vec{x}_{k0}^{T}\,\mathbf{M}\,\vec{x}_0 = a_k \vec{x}_{k0}^{T}\,\mathbf{M}\,\vec{x}_{k0}$$

and so

$$a_k = \frac{\vec{x}_{k0}^{T}\,\mathbf{M}\,\vec{x}_0}{\vec{x}_{k0}^{T}\,\mathbf{M}\,\vec{x}_{k0}}.$$

This is analogous to $A_x = \left(\hat{\imath}\cdot\vec{A}\right)/(\hat{\imath}\cdot\hat{\imath})$, where $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$. Similarly,

$$b_k = \frac{1}{\omega_k}\frac{\vec{x}_{k0}^{T}\,\mathbf{M}\,\vec{v}_0}{\vec{x}_{k0}^{T}\,\mathbf{M}\,\vec{x}_{k0}},$$

and the solution to (5) is

$$\vec{x}(t) = \sum_{j=1}^{n} \frac{\vec{x}_{j0}}{\vec{x}_{j0}^{T}\,\mathbf{M}\,\vec{x}_{j0}} \left(\vec{x}_{j0}^{T}\,\mathbf{M}\,\vec{x}_0\,\cos\omega_j t + \frac{1}{\omega_j}\vec{x}_{j0}^{T}\,\mathbf{M}\,\vec{v}_0\,\sin\omega_j t\right).$$

(The situation where $\omega_j = 0$ for some j is not hard to incorporate, and is often made part of an assignment.) This may look like a mess, but consider; once we find the $\vec{x}_{k0}$ and ω_k, and compute $\mathbf{M}\,\vec{x}_0$ and $\mathbf{M}\,\vec{v}_0$, *that's it!*.

Well, what's the point? Consider now a *driven* system,

$$\mathbf{M}\,\ddot{\vec{x}} = \mathbf{K}\,\vec{x} + \vec{F}(t),$$

where $\vec{F}$ is a column vector representing the external forces. Then, look for

$$\vec{x}(t) = \sum_{k=1}^{n} g_k(t)\,\vec{x}_{k0}, \quad \ddot{\vec{x}} = \sum_{k=1}^{n} \ddot{g}_k\,\vec{x}_{k0},$$

where the $\vec{x}_{k0}$ are the vectors found previously. Then,

$$\mathbf{M}\,\ddot{\vec{x}} = \sum_{k=1}^{n} g_k(t)\,\mathbf{M}\,\vec{x}_{k0} = \sum_{k=1}^{n} g_k\,\mathbf{K}\,\vec{x}_{k0} + \vec{F}(t).$$

But, remember that $\mathbf{K}\,\vec{x}_{k0} = -\omega_k^2\,\mathbf{M}\,\vec{x}_{k0}$, so

$$\sum_{k=1}^{n} \left(\ddot{g}_k + \omega_k^2 g_k\right)\,\mathbf{M}\,\vec{x}_{k0} = \vec{F}(t),$$

and, as before,

$$\sum_{k=1}^{n} \left(\ddot{g}_k + \omega_k^2 g_k\right) \vec{x}_{j0}^T \,\mathbf{M}\, \vec{x}_{k0} = \vec{x}_{j0}^T \,\vec{F} = \vec{x}_{j0} \cdot \vec{F}.$$

But the sum vanishes except for the $j = k$ term; then,

$$\ddot{g}_k + \omega_k^2 g_k = \frac{1}{\vec{x}_{k0}^T \,\mathbf{M}\, \vec{x}_{k0}} \vec{x}_{k0} \cdot \vec{F}(t).$$

The intial conditions for g_k are similarly found to be

$$g_k(0) = \frac{\vec{x}_{k0}^T \,\mathbf{M}\, \vec{x}_0}{\vec{x}_{k0}^T \,\mathbf{M}\, \vec{x}_{k0}}, \quad \dot{g}_k(0) = \frac{\vec{x}_{k0}^T \,\mathbf{M}\, \vec{v}_0}{\vec{x}_{k0}^T \,\mathbf{M}\, \vec{x}_{k0}}.$$

This is a *second* order *in*homogeneous equation. The solution to the homogeneous part is well known; the complete solution depends, of course, on $\vec{F}$, and may be found by variation of parameters, undetermined coefficients, annihilators, Green's functions, Laplace transforms or Tarot cards.

The main point here is; if *any* component of $\vec{F}$ is sinusoidal with *any* frequency ω_k (unless $\vec{x}_{k0} \cdot \vec{F}$ vanishes), there will be one normal mode of the system that will be driven at resonance. This is when interesting things happen.