Normality of Modes of Discrete Coupled Oscillators

As you recall, we rederived French's result as given in Equation 5-26 on page 141,

$$A_{p,n} = C_n \sin\left(\frac{pn\pi}{N+1}\right).$$

We have seen that for the N independent modes, we may restrict n to be any integer from 1 to N. What we wish to show is that these relative amplitudes satisfy our criterion for normality,

$$\sum_{p=1}^{N} A_{p,n}\, A_{p,m} = 0$$

if $n \neq m$. This necessary result actually follows readily upon use of tricks & trigonometry. To show orthogonality, we may take $C_n = 1$, so consider

$$\begin{aligned}
\sum_{p=1}^{N} A_{p,n}\, A_{p,m} &= \sum_{p=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right) \sin\left(\frac{pm\pi}{N+1}\right) \\
&= \frac{1}{2}\sum_{p=1}^{N}\left[\cos\left(\frac{n+m}{N+1}\pi p\right) - \cos\left(\frac{n-m}{N+1}\pi p\right)\right] \\
&= \frac{1}{2}\sum_{p=1}^{N}\left[\Re\left\{e^{i\frac{n+m}{N+1}\pi p}\right\} - \Re\left\{e^{i\frac{n-m}{N+1}\pi p}\right\}\right] \\
&= \frac{1}{2}\Re\left\{\sum_{p=1}^{N}\left(e^{i\frac{n+m}{N+1}\pi}\right)^p - \sum_{p=1}^{N}\left(e^{i\frac{n-m}{N+1}\pi}\right)^p\right\}.
\end{aligned}$$

Note the fancy "$\Re$" to denote "real part of". The two sums are geometric series; we know that for any α, real or imaginary,

$$\sum_{p=1}^{N} \alpha^p = \begin{cases} \left(\alpha^{N+1} - \alpha\right)/\left(\alpha - 1\right), & \alpha \neq 1, \\ N & \alpha = 1. \end{cases}$$

For this situation, let

$$\alpha_+ = e^{i\frac{n+m}{N+1}\pi}, \quad \alpha_- = e^{i\frac{n-m}{N+1}\pi}.$$

If $n+m$ is even, $n-m$ is even, and $\alpha_+^{N+1} = \alpha_-^{N+1} = 1$, so

$$\sum_{p=1}^{N} \alpha_+^p = \sum_{p=1}^{N} \alpha_-^p = -1,$$

and the quantity in braces vanishes. If $n+m$ and $n-m$ are odd, $\alpha_+^{N+1} = \alpha_-^{N+1} = -1$, and

$$\sum_{p=1}^{N} \alpha_+^p = \frac{-1-\alpha_+}{\alpha_+ - 1}, \quad \sum_{p=1}^{N} \alpha_-^p = \frac{-1-\alpha_-}{\alpha_- - 1}.$$

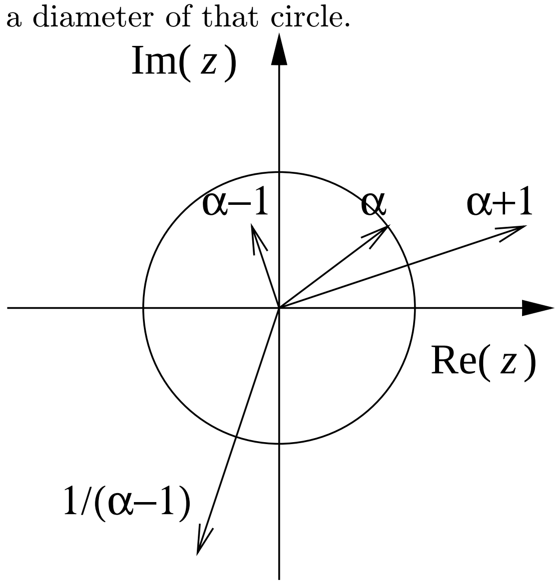
These terms are actually quite easily handled; anticipating a result that we will see in B&B Chapter 8, a more useful method to show that if $\alpha = e^{i\theta}$, with θ real, the ratio $(\alpha+1)/(\alpha-1)$ is imaginary. To do so, note that

$$\begin{aligned}\frac{\alpha+1}{\alpha-1} = \frac{e^{i\theta}+1}{e^{i\theta}-1} &= \frac{e^{-i\theta/2}}{e^{-i\theta/2}} \cdot \frac{e^{i\theta}+1}{e^{i\theta}-1} = \frac{e^{i\theta/2}+e^{-i\theta/2}}{e^{i\theta/2}-e^{-i\theta/2}} \\ &= \frac{\cos(\theta/2)}{i\sin(\theta/2)} = -i\cot(\theta/2).\end{aligned}$$

At this point, note that the case $\alpha = 1$, so that $\theta = 0$ or 2π (or any integer multiple of 2π) has been precluded, so $\sin\theta/2 = 0$ won't happen. Also, since θ is real, $\tan\theta/2$ is real, and our needed result is

$$\Re\left\{\frac{\alpha+1}{\alpha-1}\right\} = 0.$$

The above results, showing that $(\alpha+1)/(\alpha-1)$ is imaginary if $\alpha\alpha^* = 1$, may be seen graphically by plotting the pertinent complex quantities in the complex plane, as shown (the circle is the unit circle). The only tricky part, and it's not that tricky, is to see that the vectors representing $\alpha+1$ and $\alpha-1$ are perpendicular. This is seen by noting that the origin and the endpoints of $\alpha+1$ and $\alpha-1$ lie on a circle of unit radius centered at the endpoint of α, and that the vector joining the endpoints of $\alpha+1$ and $\alpha-1$ is a diameter of that circle.



The orthogonality of the modes may also be demonstrated algebraically, starting with

$$\left(\omega^2 - 2\omega_0^2\right) A_p + \omega_0^2 \left(A_{p+1} + A_{p-1}\right) = 0,$$

where $A_0 = A_{N+1} = 0$.

Consider the above set of equations for two different modes:

$$\left(\omega_m^2 - 2\omega_0^2\right) A_{p,m} + \omega_0^2 \left(A_{p+1,m} + A_{p-1,m}\right) = 0$$
$$\left(\omega_n^2 - 2\omega_0^2\right) A_{p,n} + \omega_0^2 \left(A_{p+1,n} + A_{p-1,n}\right) = 0.$$

Multiplying the first of these equations by $A_{p.n}$, the second by $A_{p,m}$ and summing over p gives

$$\sum_{p=1}^{N} \left[\left(\omega_m^2 - 2\omega_0^2\right) A_{p,m} A_{p,n} + \omega_0^2 \left(A_{p+1,m} A_{p,n} + A_{p-1,m} A_{p,n}\right)\right] = 0$$
$$\sum_{p=1}^{N} \left[\left(\omega_n^2 - 2\omega_0^2\right) A_{p,n} A_{p,m} + \omega_0^2 \left(A_{p+1,n} A_{p,m} + A_{p-1,n} A_{p,m}\right)\right] = 0,$$

and subtracting the equations and rearranging gives

$$\sum_{p=1}^{N} A_{p,m} A_{p,n} \left(\omega_m^2 - \omega_n^2\right) +$$
$$\sum_{p=1}^{N} \left(A_{p+1,m} A_{p,n} - A_{p-1,n} A_{p,m}\right) \omega_0^2 +$$
$$\sum_{p=1}^{N} \left(A_{p-1,m} A_{p,n} - A_{p+1,n} A_{p,m}\right) \omega_0^2 = 0.$$

Now, note that since $A_{0,m} = A_{0,n} = A_{N+1,m} = A_{N+1,n} = 0$,

$$\sum_{p=1}^{N} A_{p+1,m} A_{p,n} = A_{2,m} A_{1,n} + A_{3,m} A_{2,n} + \ldots + A_{N-1,m} A_{N-2,n} + A_{N,m} A_{N-1,n}$$
$$= \sum_{p=1}^{N} A_{p,m} A_{p-1,n}.$$

Similarly,

$$\sum_{p=1}^{N} A_{p-1,m} A_{p,n} = \sum_{p=1}^{N} A_{p,m} A_{p+1,n},$$

leaving

$$\sum_{p=1}^{N} A_{p,m} A_{p,n} \left(\omega_m^2 - \omega_n^2 \right) = 0.$$

If the system has no degenerate modes (*i.e.*, each mode has a different frequency), the modes are orthogonal.

The above is a special case, but not too special, of a result from linear algebra that is often used by physicists without apology, that “the eigenvectors of a Hermitian operator corresponding to distinct eigenvalues are orthogonal”. In this case, the Hermitian operator would be the matrix corresponding to the original equations, and this matrix would be Hermitian because its elements are real and the matrix is symmetric. The symmetry appears because the terms A_{p+1} and A_{p-1} appear with the same coefficient of ω_0^2, and this coefficient is the same in all of the equations. (The fact that it is the same in all equations is a manifestation of Newton’s second law.)

A more general (but not completely general) situtation is discussed in the notes **Coupled Linear Oscillators**.