18.03 Problem Set 4

(due Friday, March 22, 4:00 PM)

Part I (20 points)

HAND IN ONLY THE UNDERLINED PROBLEMS
(The others are *some* suggested choices for more practice.)
EP = Edwards & Penney; SN = Supplementary Notes (all have solutions)

Variation of parameters method. Applications
Reading: E&P §§2.5 (pp.153-156) and 2.6
Exercises:
EP §2.5 (pp.156-157) 53, 55, 57
EP §2.6 (pp.167-169) 7, 11, 17
SN §E 2D (p.4) 1a<u>bc</u>, <u>2</u> (can use eqn.(33) p.155); 2C (p.4) 10c; 2F (p.6) <u>4</u>

Fourier Series – basics
Reading: E & P §§8.1 and 8.2 [Note: *all* answers are in the back]
Exercises:
EP §8.1 (566-567) 15, 16, 21, 22
EP §8.2 (574-575) 4, 7, <u>12</u>, 13, 14
SN §7A (p.1) 1a<u>bc</u>, 2a<u>b</u>, 3, 4

Sine and cosine series. Applications to ODE's
Reading: E & P §§8.3 (pp.575-581) and 8.4
Exercises:
EP §8.3 (pp.585-586) 1, 3, 7
EP §8.4 (pp.591-592) 1, 5, 9, 13
SN §E 7B (p.1) 1<u>ab</u>, 3, 4; 7C (p.2) 1<u>a</u>b<u>c</u>, <u>2</u>, 3<u>ab</u> (use known F.S.)

DE's to ODE's by separation of variables: BVP's
Reading: E & P §§8.3 (pp.581-583), 2.8 (pp.177-179) and 8.6 (pp.606-612)
Exercises:
EP §8.3 (pp.585-586) 13
EP §2.8 (pp.189-190) 1, 3, 5
SN §E 7B (p.1) 2<u>ab</u>

Continuation. Solution of the Wave Equation
Reading: E & P §8.6
Exercises:
EP §8.6 (pp.618-620) <u>1</u>, <u>3</u>, <u>6</u>, <u>7</u>, 18 (use known F.S. whenever possible)

Part II (35 points)

Directions: Try each problem alone for 20 minutes. If you collaborate later, you must write up solutions independently. Consulting old problem sets is not permitted.

Problem 1 (3)
A certain mathematics instructor points out that the *same* linear CC differential operator $L = p(D) = \alpha D^2 + \beta D + \gamma$ is used in modeling the general behavior – and thus in particular also the behavior under sinusoidal forcing – of two types of mechanical systems and one type of electrical system: a spring-mass system, an unbalanced rotating mass, and an 'RLC' circuit. For the mechanical systems, the values of the constants (or 'system parameters') α, β and γ are taken as the mass, damping coefficient and spring constant, called in E&P m, c and k respectively; and the the relevant DE's for the steady-state displacement x_p are given in §2.6 eqn.(17) and problem # 28. For the RLC circuit, the substitutions are α, β and γ equal to L, R and $\frac{1}{C}$, where L, R and C are respectively the inductance, resistance and capacitance of the circuit; and the relevant DE for the current I is given in §2.7, eqn.(6).

A certain student questions how this can be, on the following grounds. Observe that when one uses calculus to find the angular frequency ω_m of the sinusoidal driver that will produce the maximum 'gain' or system response (called "near resonance" in the lightly-damped case), one gets different results in the each of the three cases. If one calls ω_0 the natural frequency of oscillation of the corresponding undamped (and unforced) system – i.e. the one with the same α and γ but with $\beta = 0$ in each case – then, under suitable light-damping conditions, one gets that $\omega_m < \omega_0$ for the spring-mass system, $\omega_m > \omega_0$ for the unbalanced rotating mass, and $\omega_m = \omega_0$ for the RLC circuit. (See §2.6 #27 and #28, and §2.7 eqn.(17) ff. for the analysis in each case.)
Explain how it is possible for both this (mythical) instructor and student to be correct here. (Hint: compare each of the DE's carefully, and remember that the maximum is being taken over all ω.)

Problem 2 (4: 1,2,1)
Let $L = D + P(t)$ be a first-order linear differential operator, with $P(t)$ a continuous function.
a) Using the separable method, show that the general solution to the homogeneous (in the linear sense) DE $Lx = 0$ is $x_c = c_1 x_1$, where $x_1(t) = e^{-\int P(t)\,dt}$.
b) Show that the variation-of-parameters method also works (much more easily, of course) for the case of inhomogeneous first-order equations, as follows. Take a trial solution x_p of the form $u_1(t)x_1(t)$, with x_1 a solution to $Lx = 0$, plug this trial function x_p into the DE $Lx_p = f$, and solve for the unknown function u_1, to obtain
$x_p(t) = x_1(t) \int \frac{f(t)}{x_1(t)}\,dt.$ (Note: one could interpret "$W = \det(x_1)$" $= x_1$ here.)
c) Show that combining the results of parts(a) and (b) gives the integral formula for the solution of the general first-order linear DE, eqn.(6) in §1.5.

Problem 3 (3: 1,1,1)
Let the function $f(t)$ be defined on the interval $[0, 2\pi]$ as follows:

$$f(t) = \begin{cases} t & 0 \le t \le \frac{\pi}{2} \\ \pi - t & \frac{\pi}{2} \le t \le \pi \\ 0 & \pi \le t \le 2\pi \end{cases}$$

Sketch the graphs of the following functions over at least three complete periods:

a) $\tilde{f}(t) :=$ the periodic extension of $f(t)$ with period 2π.

b) $\tilde{f}_e(t) :=$ the *even* periodic extension of $f(t)$ with period 4π.

c) $\tilde{f}_o(t) :=$ the *odd* periodic extension of $f(t)$ with period 4π.

Problem 4 (5: 3,2)
a) Let $f(t)$ be a piece-wise continuous *real*-valued periodic function with period $= 1$. Suppose that

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, e^{2\pi i n t}$$

is the Fourier series expansion of f in complex form. Use the fact that f is real-valued to express the coefficients a_n and b_n in terms of α_n and α_{-n} , where a_n and b_n denote as usual the coefficients of $\cos(2\pi nt)$ and $\sin(2\pi nt)$ respectively for the Fourier series of f expressed in real (trigonometric) form. [Use the fact that cos is even and sin is odd.]

b) Use the complex form of the Fourier series to compute the *real* Fourier series of the periodic function with period 1 defined by $f(t) = e^{at}$ on $[0, 1]$.

Note: the *complex* Fourier coefficients α_n are given by $\alpha_n = \frac{1}{2\pi} \int_0^1 f(t) \, e^{-2\pi i n t} \, dt$.

Problem 5 (10: 3,2,3,2)

The goal of this problem is to use Fourier series to solve and interpret the results for two different linear CC DE's of the form $Lx = f$ in the case where $f(t)$ is the 'square wave' function

$$f(t) = 2 \quad \text{if} \quad \left[\frac{t}{T}\right] \quad \text{is even}; \qquad f(t) = 0 \quad \text{if} \quad \left[\frac{t}{T}\right] \quad \text{is odd}$$

where $[w]$ denotes the integer part of (a real number) w and T is a given constant.

a) Using the known Fourier series for $f(t)$, solve the DE $x' + kx = f(t)$ for the steady-state solution $x_p(t)$ in Fourier series form ($k > 0$ denoting a given constant).

b) Taking the specific values $k = 1$ and $T = 2$, approximate x_p by its Fourier series up to and including $n = 25$; then graph the input function $f(t)$ and this ($n = 25$) approximation to the steady-state system response $x_p(t)$ side-by-side over at least three periods. (Note: it may be easier to graph the square wave $f(t)$ by hand.)

If this DE is modeling a one-tank mixing situation, what is the response of the system to a periodic on-and-off constant input? Discuss in terms of the period, amplitude and lag observed in the steady-state response x_p.

c) Repeat the analysis above for the DE $x'' + kx = f(t)$. For the graphing component, again take $k = 1$ and $T = 2$. For the real-world interpretation, use the spring-mass system. (In this case, $n = 5$ will do.)

d) For the case $Lx = x'' + kx$ with $k = 1$: find the value(s) of T which will produce the largest system response x_p.

Problem 6 (10+: 1,4,4,1; EC)

The goal of this problem is to find the Fourier and d'Alembert forms of the solution to the wave equation under following the end-conditions: instead of the ends of the wire being fixed for all time, both ends are allowed to slide (without friction) vertically up and down from their starting positions. We can visualize this situation, for example, by imagining that each end of the wire goes into a little metal sleeve and is secured with a button, and that these sleeves run in (well-lubricated) vertical tracks. The sleeves are there to force the ends to satisfy the boundary condition that each end is always *perpendicular* to its vertical track at all times.

a) Modify the set up for the situation of one free and one clamped end given in E&P §8.6 #18 for this situation, in which *both* ends are allowed to slide vertically. Take the IC's as in #18: an initial displacement of the form $f(x)$, but no initial velocity.

b) Use separation-of-variables to solve for the subsequent displacement of the wire $y(x, t)$ in Fourier form.

c) Use the Fourier form found in part(b) to solve for the subsequent displacement of the wire $y(x, t)$ in d'Alembert form.

d) For an initial displacement $f(x)$ of the form of a plucked string, as in Example 2 in §8.6, what would you predict that the wire would do over time? (Remember that there's no damping in this idealized situation.)

e) EC: Take $L = \pi$ and a center height of 1 for intial plucked displacement, graph a few d'Alembert 'snapshots' for selected positive times, and see what the theory predicts for the motion of the wire.