Normal Modes of Coupled Pendulums

In Chapter 5, French considers masses coupled by springs or by a string under tension, with the masses at the ends attached to a fixed points. For N oscillators the results obtained are

$$\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \qquad (5-25)$$

for the frequencies, where $\omega_0 = \sqrt{k/m}$ for masses connected by springs of spring constant k or $\omega_0 = \sqrt{T/ml}$ for masses separated by a distance l on a string with tension T and

$$A_{p,n} = C_n \sin\left(\frac{pn\pi}{N+1}\right) \qquad (5-26)$$

as the displacement amplitudes of the masses oscillating in mode n.

For the case of pendulums coupled by springs, as dicussed in the notes from previous years, the equations in the amplitudes and frequencies that result from the equations of motion are

$$\begin{aligned}
\left(-\omega^2 + \omega_p^2 + \omega_s^2\right) A_1 - \omega_s^2 A_2 &= 0 \\
\left(-\omega^2 + \omega_p^2 + 2\omega_s^2\right) A_2 - \omega_s^2 \left(A_3 + A_1\right) &= 0 \\
&\vdots \\
\left(-\omega^2 + \omega_p^2 + 2\omega_s^2\right) A_p - \omega_s^2 \left(A_{p+1} + A_{p-1}\right) &= 0 \\
&\vdots \\
\left(-\omega^2 + \omega_p^2 + \omega_s^2\right) A_N - \omega_s^2 A_{N-1} &= 0.
\end{aligned}$$

This should be compared to the equations on Page 139 (note the typo in the last equation in the text). The crucial distinction is in the first and last equations, where there is no factor of 2 multiplying ω_s^2 in the A_1 and A_N terms. This means that even if we set the now artificial quantities A_0 and A_{N+1} to zero, we cannot get a single equation equivalent to Eq. 5-18. What we get are three equations:

$$\left(-\omega^2 + \omega_p^2 + 2\omega_s^2\right) A_p - \omega_s^2 \left(A_{p+1} + A_{p-1}\right) = 0$$

for $p = 2, \ldots N-1$ and

$$\begin{aligned}
\left(-\omega^2 + \omega_p^2 + \omega_s^2\right) A_1 - \omega_s^2 A_2 &= 0 \\
\left(-\omega^2 + \omega_p^2 + \omega_s^2\right) A_N - \omega_s^2 A_{N-1} &= 0.
\end{aligned}$$

It would be advantageous to find formulae similar to Equations (5-25) and (5-26), and it turns out that there are such forms. The derivation is not as simple, and will not be included here. The needed expressions are

$$\omega^2 = \omega_p^2 + 4\omega_s^2 \sin^2\left(\frac{n\pi}{N}\right)$$

$$A_{p,n} = C_n \cos\left(\frac{(2p-1)n\pi}{2N}\right)$$

where $n = 0 \ldots N-1$ and $p = 1 \ldots N$.

In the above, $n = 0$ gives a non-trivial solution with $\omega_0 = \omega_p$ and $A_{p,n} = C_n$ for all p. This is expected, as this lowest frequency mode corresponds to all of the pendulums swinging together in phase, with the coupling springs unstretched. This mode does not correspond to anything in the situation where the masses at the ends are attached to a fixed point.

Using the same method outlined in **Normality of Modes of Discrete Coupled Oscillators**, it can be shown that

$$\sum_{p=1}^{N} A_{p,m} A_{p,n} = 0$$

if $m \neq n$. Specifically, if $m = 0$, it follows that

$$\sum_{p=1}^{N} A_{p,n} = 0$$

for $n \neq 0$, as can be checked explicitly for $n = 1, 2, 3$ with the amplitudes as given in the prior notes.