More on Cyclotron Fields

Equation 1.128 in B&B, and hence Eq. 1.129, is not as general as it could be. Arbitrary electric fields can be considered.

First, some notational things. I'd like to use t' as a dummy integration variable. This is a situation where the circumstances will be needed to recognize that t' is a dummy variable, not a derivative. Also, I'd like to allow for the possibility that the electric field has both an x- and a y-component, $\vec{\boldsymbol{E}} = E_x\,\hat{x} + E_y\,\hat{y}$, and then let $\vec{\epsilon} \equiv \frac{q}{m}\vec{\boldsymbol{E}}$. Recall that ω_{c}, the cyclotron frequency, is $\frac{qB_0}{m}$. The equations of motion are then

$$\dot{v}_x = \epsilon_x + \omega_{\mathrm{c}} v_y, \quad \dot{v}_y = \epsilon_y - \omega_{\mathrm{c}} v_x.$$

Denote the initial values of the velocity components (which we won't really need, but let's be complete) as $v_x(0) = v_{x0}$, $v_y(0) = v_{y0}$.

Now, similar to what was done in the text, by differentiating the first equation and substituting into the second we get a single equation, second order, for v_x;

$$\ddot{v}_x + \omega_{\mathrm{c}}^2 v_x = \dot{\epsilon}_x + \omega_{\mathrm{c}}\epsilon_y.$$

The solution, which may be found any number of ways (see any differential equations text, or ask), is

$$\begin{aligned} v_x = {} & \frac{1}{\omega_{\mathrm{c}}}\int_0^t \sin(\omega_{\mathrm{c}} t - \omega_{\mathrm{c}} t')\,\dot{\epsilon}_x(t')\,dt' \\ & + \int_0^t \sin(\omega_{\mathrm{c}} t - \omega_{\mathrm{c}} t')\,\epsilon_y(t')\,dt' \\ & + v_{x0}\cos\omega_{\mathrm{c}} t + \left(\frac{\epsilon_{x0}}{\omega_{\mathrm{c}}} + v_{y0}\right)\sin\omega_{\mathrm{c}} t. \end{aligned}$$

Note that in the last term, $\dot{v}_x(0) = a_{x0} = \epsilon_x(0) + \omega_c v_{y0}$.

The first term may be integrated by parts;

$$\begin{aligned} & \frac{1}{\omega_{\mathrm{c}}}\int_0^t \sin(\omega_{\mathrm{c}} t - \omega_{\mathrm{c}} t')\,\dot{\epsilon} x(t')\,dt' \\ & \qquad = \frac{1}{\omega_{\mathrm{c}}}\sin(\omega_{\mathrm{c}} t - \omega_{\mathrm{c}} t')\,\epsilon_x(t')\Big|_0^t + \int_0^t \cos(\omega_{\mathrm{c}} t - \omega_{\mathrm{c}} t')\,\epsilon_y(t')\,dt \\ & \qquad = -\frac{1}{\omega_{\mathrm{c}}}(\sin\omega_{\mathrm{c}} t)\,\epsilon_x(0) + \int_0^t \cos(\omega_{\mathrm{c}} t - \omega_{\mathrm{c}} t')\,\epsilon_y(t')\,dt, \end{aligned}$$

so that the result is

$$v_x = \int_0^t \left[\cos(\omega_c t - \omega_c t')\, \epsilon_x(t') + \sin(\omega_c t - \omega_c t')\, \epsilon_y(t')\right] dt' \\ + v_{x0} \cos \omega_c t + v_{y0} \sin \omega_c t.$$

A similar result holds for v_y.

The above may be summarized very cleanly by use of matrix calculus. Sparing the details (although you are encouraged to check),

$$\vec{v} = \int_0^t \mathrm{M}(\omega_c t - \omega_c t')\, \vec{\epsilon}(t')\, dt' + \mathrm{M}(\omega_c t)\vec{v}_0,$$

where M is a matrix,

$$\mathrm{M}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

The matrix $\mathrm{M}(\theta)$ has many amazing properties. It can be shown that

$$\mathrm{M}(\theta) = \exp\left(\mathrm{R}\theta\right), \quad \text{where} \quad \mathrm{R} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and the exponentiation of a matrix is done *via* a series expansion. To be amazed, note that the eigenvalues of M and R are $\pm i$, $\mathrm{M}(0) = \mathrm{I}$, the identity matrix, $\mathrm{M}(\pi/2) = \mathrm{R}$, $\frac{d\mathrm{M}}{d\theta} = \mathrm{RM}$, and $\mathrm{M}(\alpha)\mathrm{M}(\beta) = \mathrm{M}(\alpha + \beta)$. M rotates a vector by the angle θ, and R is known as the generator of rotations in the plane.