Supplement to Supplementary Notes on Fourier Transforms

(This material is being given separately, because I'm not sure if it's too much to include with the already long Notes on Fourier Transforms. Your feedback is appreciated.)

Consider the Wavy Quation,

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2},$$

with boundary conditions at $t = 0$,

$$\psi(x,\,0) = f(x), \quad \frac{\partial\psi}{\partial t}\Big|_{t=0} = g(x).$$

Denote the Fourier transforms of $f(x)$ and $g(x)$ as $F(k)$ and $G(k)$, and

$$\Psi(k,\,t) = \int_{-\infty}^{\infty} \psi(x,\,t)\, e^{ikx}\, dx,$$

so that $F(k) = \Psi(k,\,0)$, $G(k) = \frac{\partial\Psi}{\partial t}|_{t=0}$. At this point, it shouldn't hurt to recall the inverse transform

$$\psi(x,\,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(k,\,t)\, e^{-ikx}\, dk$$

and the δ-distribution

$$\delta(x' - x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik(x-x')}\, dk.$$

As it turns out, we will be using some properties of the δ-distribution that are common to functions.

Anyhow, as was done with the Heaty Quation,

$$\frac{1}{v^2}\int_{-\infty}^{\infty} \frac{\partial^2\psi}{\partial t^2}\, e^{ikx}\, dx = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}, \quad \int_{-\infty}^{\infty} \frac{\partial^2\psi}{\partial x^2}\, e^{ikx}\, dx = -k^2\Psi,$$

so that

$$-k^2\Psi = \frac{1}{v^2}\frac{\partial\Psi}{\partial t^2}. \tag{1}$$

This is readily solved, and we can and will write the solution as

$$\Psi(k,\,t) = \Psi_+(k)e^{ikvt} + \Psi_-(k)e^{-ikvt}.$$

The boundary conditions at $t = 0$ give

$$\Psi_+(k) + \Psi_-(k) = F(k), \quad ikv\left(\Psi_+(k) - \Psi_-(k)\right) = G(k),$$

which are solved for

$$\Psi_+(k) = \frac{1}{2}\left(F(k) + \frac{1}{ikv}G(k)\right), \quad \Psi_-(k) = \frac{1}{2}\left(F(k) - \frac{1}{ikv}G(k)\right),$$

and so

$$\Psi(k,\, t) = \frac{1}{2}\left(F(k) + \frac{1}{ikv}G(k)\right)\, e^{ikvt} + \frac{1}{2}\left(F(k) - \frac{1}{ikv}G(k)\right)\, e^{-ikvt}.$$

Now it's time to take the inverse transform; you knew it had to happen sometime. With all of the terms,

$$\begin{aligned}
\psi(x,\, t) &= \frac{1}{2\pi}\left[\begin{array}{l} \frac{1}{2}\int_{-\infty}^{\infty} F(k)\,\left(e^{ikvt} + e^{-ikvt}\right)\, e^{-ikx}\, dk \\ \quad + \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{ikv}\, G(k)\left(e^{ikvt} - e^{-ikvt}\right)\, e^{-ikx}\, dk \end{array}\right] \\
&= \frac{1}{2}\int_{-\infty}^{\infty} f\left(x'\right)\left[\begin{array}{l} \int_{-\infty}^{\infty} \frac{1}{2\pi}\, e^{ik(x-x'+vt)}\, dk \\ \quad + \int_{-\infty}^{\infty} \frac{1}{2\pi}\, e^{-ik(x-x'-vt)}\, dk \end{array}\right] dx' \\
&+ \frac{1}{2}\int_{-\infty}^{\infty} g\left(x'\right)\left[\begin{array}{l} \int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{1}{ikv} e^{-ik(x-x'+vt)}\, dk \\ \quad - \int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{1}{ikv}\, e^{-ik(x-x'-vt)}\, dk \end{array}\right] dx'.
\end{aligned}$$

This mess is tractable; the first integral (with $f\left(x'\right)$ in the integrand) is the sum of δ-functions. The second integral (with $g\left(x'\right)$) is also the sum of δ-functions, but cleverly disgused. Consider the integral

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{ik}\frac{1}{2\pi}\, e^{-ik(x-x')}\, dk &= \int_{-\infty}^{x'}\int_{-\infty}^{\infty} \frac{1}{2\pi}\, e^{-ik(x-x'')}\, dk\, dx'' \\
&= \int_{-\infty}^{x'} \delta(x'' - x)dx'' \\
&= \frac{1}{2}\left(\mathrm{sgn}(x' - x) + 1\right).
\end{aligned}$$

With this integral, our result is

$$\begin{aligned}\psi(x,\,t) &= \frac{1}{2}\left[f(x+vt)+f(x-vt)\right] \\ &\quad + \frac{1}{2v}\int_{-\infty}^{\infty} g(x')\frac{1}{2}\left(\operatorname{sgn}(x+vt-x') - \operatorname{sgn}(x-vt-x')\right)\,dx' \\ &= \frac{1}{2}\left[f(x+vt)+f(x-vt)\right] + \frac{1}{2v}\int_{x-vt}^{x+vt} g(x')\,dx',\end{aligned}$$

which is d'Alembert's solution.