# A Bit of Math Regarding Critical Damping

In lecture last week (February 7), in introducing critical damping, Prof. van Oudenaarden alluded to a crucial result from the mathematicians; specfically, the solutions to the differential equation

$$\ddot{x} + \gamma\,\dot{x} + \omega_0^2\,x = 0, \qquad (\clubsuit)$$

in the case where $\omega_0 = \dfrac{\gamma}{2}$ has (real) solutions of the form

$$x(t) = (A + B\,t)\;e^{-\frac{\gamma}{2}t}.$$

This result was presented without derivation, and indeed the derivation is a very useful element of any differential equations class, but not really necessary for 8.03 purposes.

A casual glance at the DE texts on my shelf (*Differential Equations: A Modeling Approach* by Borrelli & Coleman, *Elementary Differential Equations with Boundary Value Problems* by Edwards & Penney and *Elementary Differential Equations with Boundary Value Problems* by Boyce & Di Prima. Why does it always take two authors, and aren't there other title names available?) show three different ways of showing the above result. I'd like to include a fourth, more motivated by physics.

Consider an object whose motion is described by $\clubsuit$, subject to the initial conditions $x(0) = 0$, $\dot{x}(0) = v_0$, in the two cases $\omega_0 > \gamma/2$ (underdamped) and $\omega_0 < \gamma/2$ (overdamped). For the underdamped case, the solution to $\clubsuit$, with the initial conditions, can be shown (that means you do it) to be

$$x(t) = \frac{v_0}{\omega}\left(e^{-\frac{\gamma}{2}t}\right)\sin\left(\omega t\right), \qquad (\diamondsuit)$$

where $\omega^2 = \omega_0^2 - \left(\gamma/2\right)^2$ ($\omega$ is real for the underdamped case).

Now, let's consider what happens when $\omega$ approaches zero (that is, when $\omega_0$ approaches $\gamma/2$). Rewrite ($\diamondsuit$) as

$$x(t) = v_0\,t\left(e^{-\frac{\gamma}{2}t}\right)\frac{\sin(\omega t)}{(\omega t)}. \qquad (\diamondsuit')$$

Here's the tricky part, and it's not too tricky: For any time $t$, the product $\omega t$ will approach zero as $\omega$ approaches zero. If we wanted to be more mathematically rigorous (and we don't), we would need to address "uniform convergence;" check *Principles of Mathematical Analysis* by Walter Rudin (that's the 18.100B text) for details. So, let's give the product $\omega t$ a name, for our purposes $\epsilon$; in the limit as $\omega$ approaches zero, we have

$$\begin{aligned} x(t) &= v_0 \left( t\, e^{-\frac{\gamma}{2}t} \right) \frac{\sin(\epsilon)}{\epsilon} \\ &\to v_0\, t\, e^{-\frac{\gamma}{2}t}, \end{aligned}$$

where the standard result

$$\lim_{\epsilon \to 0} \frac{\sin(\epsilon)}{\epsilon} = 1 \qquad (\heartsuit)$$

has been used.

The result given in ($\heartsuit$) may be shown in many ways. If you used *Calculus and Analytic Geometry* by George Simmons, a geometric derivation is on Pages 71-72. If you've seen and appreciated l'Hôpital's Rule, ($\heartsuit$) follows from a simple (really, I get to use the word "simple") application. From the Taylor Series for $\sin(\epsilon)$,

$$\frac{\sin(\epsilon)}{\epsilon} = 1 - \frac{\epsilon^2}{3!} + \frac{\epsilon^4}{5!} + \cdots,$$

from which ($\heartsuit$) follows without apology.

For the overdamped case, $\gamma/2 > \omega_0$, so $\omega$ would be imaginary. Although we could deal with $\omega$ imaginary, that would be more math than physics (see below). In any event, it takes a few steps to show that

$$x(t) = \frac{v_0}{2\,\xi} \left( -e^{-\left(\frac{\gamma}{2} + \xi\right)\, t} + e^{-\left(\frac{\gamma}{2} - \xi\right)\, t} \right) = \frac{v_0}{\xi} \left( e^{-\frac{\gamma}{2}\, t} \right) \sinh(\xi\, t),$$

where $\xi$, the unprouncable Greek lowercase "xi", is $\xi = \sqrt{(\gamma/2)^2 - \omega_0^2}$. For the overdamped case, $\xi$ is real.

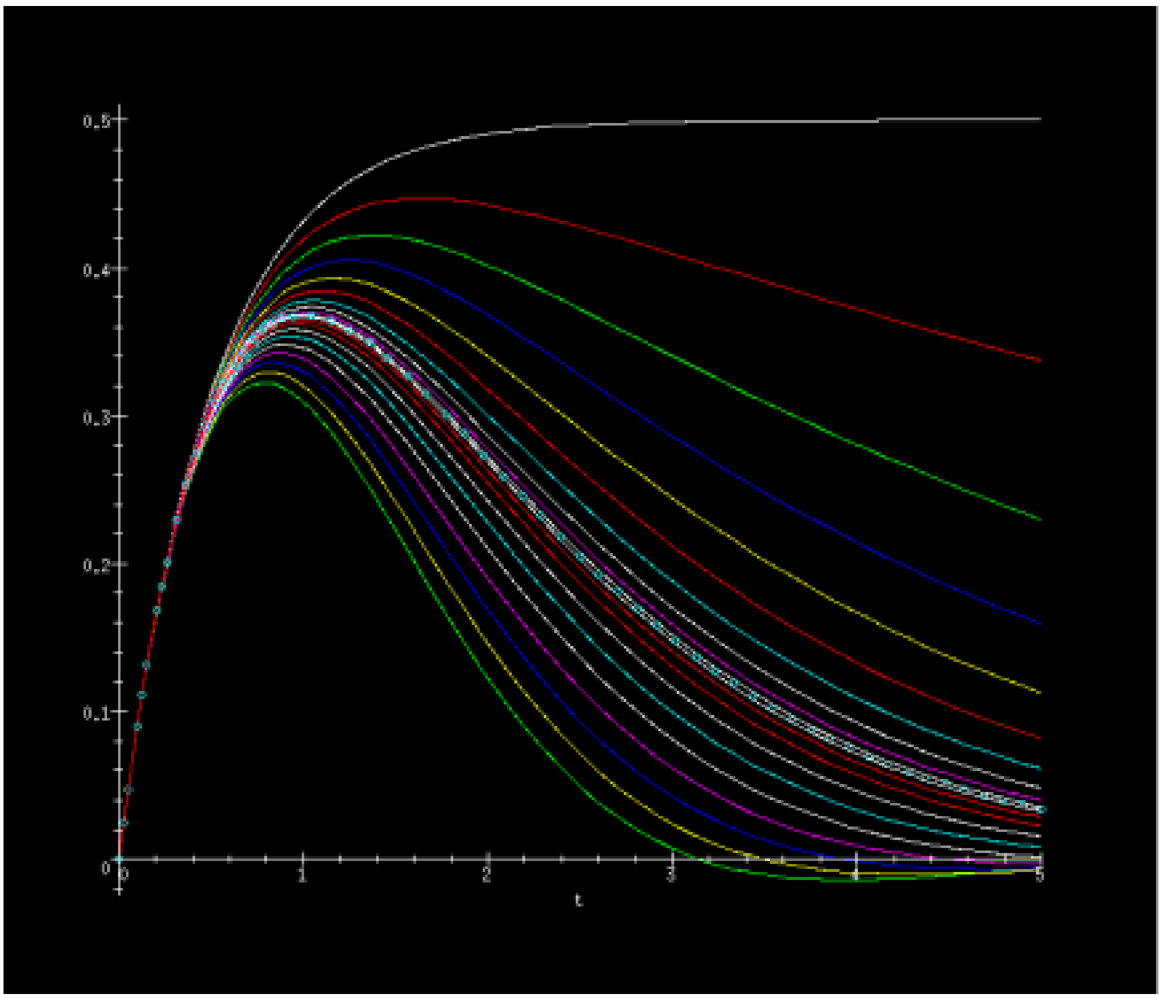
By the same methods as above (but the geometric exposition is far trickier), it can be shown that

$$\lim_{\xi \to 0} \frac{\sinh(\xi\, t)}{\xi\, t} = 1, \qquad (\spadesuit)$$

reproducing

$$x(t) = v_0\, t\, e^{-\frac{\gamma}{2}t}.$$

The graphs below are plots of $x(t)$ for different values of $\epsilon$ or $\eta = \xi\, t$, scaled to $\gamma = 2$ and $v_0 = 1$. The plot that has the circles is $x(t) = t\, e^{-t}$, the limit as $\epsilon = \eta = 0$. The reason the circled plot are hard to see is that the curves for $\epsilon = \eta = 1/10$ is so close to the limiting case that the curves become indistinguishable, which is sort of the whole point of doing this.




Some of you have seen that $\sin(j\epsilon) = j\ \sinh(\epsilon)$, which is completely consistent with the above result. This is typical of much of what happens in physics; get used to it.