

## Supplemental Notes

## A Bit of Math Regarding Critical Damping

In lecture last week (February 7), in introducing critical damping, Prof. van Oudenaarden alluded to a crucial result from the mathematicians; specifically, the solutions to the differential equation

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0, \quad (\clubsuit)$$

in the case where  $\omega_0 = \frac{\gamma}{2}$  has (real) solutions of the form

$$x(t) = (A + Bt) e^{-\frac{\gamma}{2}t}.$$

This result was presented without derivation, and indeed the derivation is a very useful element of any differential equations class, but not really necessary for 8.03 purposes.

A casual glance at the DE texts on my shelf (*Differential Equations: A Modeling Approach* by Borrelli & Coleman, *Elementary Differential Equations with Boundary Value Problems* by Edwards & Penney and *Elementary Differential Equations with Boundary Value Problems* by Boyce & Di Prima. Why does it always take two authors, and aren't there other title names available?) show three different ways of showing the above result. I'd like to include a fourth, more motivated by physics.

Consider an object whose motion is described by  $\clubsuit$ , subject to the initial conditions  $x(0) = 0$ ,  $\dot{x}(0) = v_0$ , in the two cases  $\omega_0 > \gamma/2$  (underdamped) and  $\omega_0 < \gamma/2$  (overdamped). For the underdamped case, the solution to  $\clubsuit$ , with the initial conditions, can be shown (that means you do it) to be

$$x(t) = \frac{v_0}{\omega} \left( e^{-\frac{\gamma}{2}t} \right) \sin(\omega t), \quad (\diamond)$$

where  $\omega^2 = \omega_0^2 - (\gamma/2)^2$  ( $\omega$  is real for the underdamped case).

Now, let's consider what happens when  $\omega$  approaches zero (that is, when  $\omega_0$  approaches  $\gamma/2$ ). Rewrite  $(\diamond)$  as

$$x(t) = v_0 t \left( e^{-\frac{\gamma}{2}t} \right) \frac{\sin(\omega t)}{(\omega t)}. \quad (\diamond')$$

Here's the tricky part, and it's not too tricky: For any time  $t$ , the product  $\omega t$  will approach zero as  $\omega$  approaches zero. If we wanted to be more mathematically rigorous (and we don't), we would need to address "uniform convergence"; check *Principles of Mathematical Analysis* by Walter Rudin (that's the 18.100B text) for details. So, let's give the product  $\omega t$  a name, for our purposes  $\epsilon$ ; in the limit as  $\omega$  approaches zero, we have

$$x(t) = v_0 \left( t e^{-\frac{\gamma}{2}t} \right) \frac{\sin(\epsilon)}{\epsilon}$$

$$\rightarrow v_0 t e^{-\frac{\gamma}{2}t},$$

where the standard result

$$\lim_{\epsilon \rightarrow 0} \frac{\sin(\epsilon)}{\epsilon} = 1 \quad (\heartsuit)$$

has been used.

The result given in  $(\heartsuit)$  may be shown in many ways. If you used *Calculus and Analytic Geometry* by George Simmons, a geometric derivation is on Pages 71-72. If you've seen and appreciated l'Hôpital's Rule,  $(\heartsuit)$  follows from a simple (really, I get to use the word "simple") application. From the Taylor Series for  $\sin(\epsilon)$ ,

$$\frac{\sin(\epsilon)}{\epsilon} = 1 - \frac{\epsilon^2}{3!} + \frac{\epsilon^4}{5!} + \dots,$$

from which  $(\heartsuit)$  follows without apology.

For the overdamped case,  $\gamma/2 > \omega_0$ , so  $\omega$  would be imaginary. Although we could deal with  $\omega$  imaginary, that would be more math than physics (see below). In any event, it takes a few steps to show that

$$x(t) = \frac{v_0}{\xi} \left( e^{-\left(\frac{\gamma}{2} + \xi\right)t} - e^{-\left(\frac{\gamma}{2} - \xi\right)t} \right) = \frac{v_0}{\xi} \left( e^{-\frac{\gamma}{2}t} \right) \sinh(\xi t),$$

where  $\xi$ , the unprouncable Greek lowercase "xi", is  $\xi = \sqrt{(\gamma/2)^2 - \omega_0^2}$ . For the overdamped case,  $\xi$  is real.

By the same methods as above (but the geometric exposition is far trickier), it can be shown that

$$\lim_{\xi \rightarrow 0} \frac{\sinh(\xi)}{\xi} = 1, \quad (\spadesuit)$$

reproducing

$$x(t) = v_0 t e^{-\frac{\gamma}{2}t}.$$

Some of you have seen that  $\sin(j\epsilon) = j \sinh(\epsilon)$ , which is completely consistent with the above result. This is typical of much of what happens in physics; get used to it.