More on Low-Temperature Fermions -  
a Reconsideration of Baierlein’s Equation (9.15)

The use of $\delta \varepsilon \sim \text{“a few } kT\text{”}$ can be made a bit better. Consider Figure 9.3 on Page 187, and take the slope of the diagonal part of the curve to be

$$\frac{df}{d\varepsilon}\bigg|_{\varepsilon=\varepsilon_F} = -\frac{1}{4kT}.$$ 

Then, a simple “rise over run” gives

$$-\frac{1}{4kT} = -\frac{1}{2\delta \varepsilon} \quad \rightarrow \quad \delta \varepsilon = 2kT;$$

the coefficient of $9/4$ in the line preceding Equation (9.15) becomes unity, quite close to $\zeta(2) = \frac{\pi^2}{6}$.

To find the variation with the exact version of the Fermi distribution $f(\varepsilon)$, replace the last two terms in Equation (9.14) with the integrals;

$$\int_{\mu(T)}^{\infty} D(\varepsilon) f(\varepsilon) \, d\varepsilon - \int_{\mu(T)}^{0} D(\varepsilon) (1 - f(\varepsilon)) \, d\varepsilon.$$

Noting that the only substantial contribution to either integral is in the region of integration very close to $\varepsilon = \mu(T)$ allows us to use a first-order Taylor Series for $D(\varepsilon)$ about $\varepsilon = \mu(T)$ and to extend the range of the second integral from $-\infty$ to $\mu(T)$. Of course, $D(\varepsilon)$ is not defined for $\varepsilon < 0$, and the Taylor Series approximation is far from valid, but $f(\varepsilon)$ is defined in this range, and will be exponentially small; we haven’t changed the value of the integral in any substantial way.

Using Baierlein’s notation, the integrals combine to give

$$D(\varepsilon_F) \left[ \int_{\mu(T)}^{\infty} f(\varepsilon) \, d\varepsilon - \int_{-\infty}^{\mu(T)} (1 - f(\varepsilon)) \, d\varepsilon \right]$$

$$+ D'(\varepsilon_F) \left[ \int_{\mu(T)}^{\infty} (\varepsilon - \mu(T)) f(\varepsilon) \, d\varepsilon - \int_{-\infty}^{\mu(T)} (\varepsilon - \mu(T)) (1 - f(\varepsilon)) \, d\varepsilon \right].$$

Make the change of variable $x = (\varepsilon - \mu(T))/kT$ to obtain

$$D(\varepsilon_F) \frac{kT}{2} \left[ \int_{0}^{\infty} \frac{dx}{e^x + 1} - \int_{-\infty}^{0} \frac{e^x \, dx}{e^x + 1} \right]$$

$$+ D'(\varepsilon_F) (kT)^2 \left[ \int_{0}^{\infty} \frac{x \, dx}{e^x + 1} - \int_{-\infty}^{0} \frac{x \, e^x \, dx}{e^x + 1} \right].$$
Make the further change of variable \(x \rightarrow -x\) in the second integral in each line above, so that all four integrals are from 0 to \(+\infty\); the first two integrals, those multiplying \(D(\varepsilon_F)\), cancel, while those in the second line add, giving a net correction of

\[
2 D'(\varepsilon_F) (kT)^2 \int_0^\infty \frac{x \, dx}{e^x + 1}.
\]

The last integral is similar to those discussed in Appendix A of the text, except for the presence of a + sign in the denominator. However, the improper integral certainly converges, and a quick consultation with MAPLE (or some other symbolic-manipulation program) does indeed give the integral as \(\pi^2/12\), which leads to Equation (9.15).

To see how this integral is obtained (this is math now - we’ve done the physics), follow the same steps as in Equation (A7), Page 422, arriving at a sum with alternating signs,

\[
\int_0^\infty \frac{x \, dx}{e^x + 1} = \left( \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2} \right) \times \int_0^\infty e^{-y} \, y \, dy = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2}.
\]

Express the sum as

\[
\sum_{n \text{ odd}} \frac{1}{n^2} - \sum_{n \text{ even}} \frac{1}{n^2} = \sum_{\text{all } n} \frac{1}{n^2} - 2 \sum_{n \text{ even}} \frac{1}{n^2} = \sum_{\text{all } n} \frac{1}{n^2} - 2 \times \frac{1}{4} \times \sum_{\text{all } n} \frac{1}{n^2} = \frac{1}{2} \sum_{\text{all } n} \frac{1}{n^2} = \frac{1}{2} \zeta(2) = \frac{\pi^2}{12}
\]

where the well-known result for \(\zeta(2)\) has been used. Of the many ways to see this sum, see, for instance, the notes Some Sums, linked from the 8.03-esg page.