More on Low-Temperature Fermions -
a Reconsideration of Baierlein's Equation (9.15)

The use of $\delta\varepsilon \sim$ "a few kT" can be made a bit better. Consider Figure 9.3 on Page 187, and take the slope of the diagonal part of the curve to be

$$\left.\frac{df}{d\varepsilon}\right|_{\varepsilon=\varepsilon_{\rm F}} = -\frac{1}{4\,kT}.$$

Then, a simple "rise over run" gives

$$-\frac{1}{4\,kT} = \frac{-1}{2\,\delta\varepsilon} \qquad \longrightarrow \qquad \delta\varepsilon = 2\,kT;$$

the coefficient of 9/4 in the line preceding Equation (9.15) becomes unity, quite close to $\zeta(2) = \frac{\pi^2}{6}$.

To find the variation with the exact version of the Fermi distribution $f(\varepsilon)$, replace the last two terms in Equation (9.14) with the integrals;

$$\int_{\mu(T)}^{\infty} D(\varepsilon) f(\varepsilon)\, d\varepsilon - \int_0^{\mu(T)} D(\varepsilon)(1 - f(\varepsilon))\, d\varepsilon.$$

Noting that the only substantial contribution to either integral is in the region of integration very close to $\varepsilon = \mu(T)$ allows us to use a first-order Taylor Series for $D(\varepsilon)$ about $\varepsilon = \mu(T)$ and to extend the range of the second integral from $-\infty$ to $\mu(T)$. Of course, $D(\varepsilon)$ is not defined for $\epsilon < 0$, and the Taylor Series approximation is far from valid, but $f(\epsilon)$ is defined in this range, and will be exponentially small; we haven't changed the value of the integral in any substantial way.

Using Baierlein's notation, the integrals combine to give

$$\begin{aligned} &D\left(\varepsilon_{\rm F}\right)\left[\int_{\mu(T)}^{\infty} f(\varepsilon)\, d\epsilon - \int_{-\infty}^{\mu(T)} (1 - f(\varepsilon))\, d\epsilon\right] \\ &+ D'(\varepsilon_{\rm F})\left[\int_{\mu(T)}^{\infty} (\varepsilon - \mu(T))\, f(\varepsilon)\, d\epsilon - \int_{-\infty}^{\mu(T)} (\varepsilon - \mu(T))\,(1 - f(\varepsilon))\, d\epsilon\right]. \end{aligned}$$

Make the change of variable $x = (\varepsilon - \mu(T))/kT$ to obtain

$$\begin{aligned} &D\left(\varepsilon_{\rm F}\right)\, kT \left[\int_0^{\infty} \frac{dx}{e^x + 1} - \int_{-\infty}^{0} \frac{e^x\, dx}{e^x + 1}\right] \\ &+ D'(\varepsilon_{\rm F})\,(kT)^2 \left[\int_0^{\infty} \frac{x\, dx}{e^x + 1} - \int_{-\infty}^{0} \frac{x\, e^x\, dx}{e^x + 1}\right]. \end{aligned}$$

Make the further change of variable $x \to -x$ in the second integral in each line above, so that all four integrals are from 0 to $+\infty$; the first two integrals, those multiplying $D\,(\varepsilon_{\mathrm{F}})$, cancel, while those in the second line add, giving a net correction of

$$2\,D'(\varepsilon_{\mathrm{F}})\,(kT)^2 \int_0^\infty \frac{x\,dx}{e^x+1}.$$

The last integral is similar to those discussed in Appendix A of the text, except for the presence of a + sign in the denominator. However, the improper integral certainly converges, and a quick consultation with MAPLE (or some other symbolic-manipulation program) does indeed give the integral as $\pi^2/12$, which leads to Equation (9.15).

To see how this integral is obtained (this is math now - we've done the physics), follow the same steps as in Equation (A7), Page 422, arriving at a sum with alternating signs,

$$\int_0^\infty \frac{x\,dx}{e^x+1} = \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}\right) \times \int_0^\infty e^{-y}\,y\,dy = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.$$

Express the sum as

$$\begin{aligned}
\sum_{n\,\mathrm{odd}} \frac{1}{n^2} - \sum_{n\,\mathrm{even}} \frac{1}{n^2} &= \sum_{\mathrm{all}\,n} \frac{1}{n^2} - 2\sum_{n\,\mathrm{even}} \frac{1}{n^2} \\
&= \sum_{\mathrm{all}\,n} \frac{1}{n^2} - 2 \times \frac{1}{4} \times \sum_{\mathrm{all}\,n} \frac{1}{n^2} \\
&= \frac{1}{2} \sum_{\mathrm{all}\,n} \frac{1}{n^2} = \frac{1}{2}\zeta(2) = \frac{\pi^2}{12}
\end{aligned}$$

where the well-known result for $\zeta(2)$ has been used. Of the many ways to see this sum, see, for instance, the notes **Some Sums**, linked from the 8.03-esg page.