Fourier Transforms
including
More Dirty Tricks
and
an Unattractive but Highly Useful Example

Let's start with B&B Problem 2.12(a), with a slight variation. In statisticians' notation,

$$N(\mu,\,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}.$$

Note that the independent variable x is not included in writing $N(\mu,\,\sigma)$; the idea is that N, a *normal distribution*, is completely characterized by the parameters μ and σ. The normal distribution has the following neat properties, any of which you can verify:

i) N is maximized at $x=\mu$,
ii) $\int N\,dx=1$,
iii) $\int x\,N\,dx=\mu$, and
iv) $\int (x-\mu)^2\,N\,dx=\sigma^2$,

where the integrals in (ii), (iii) and (iv) are over the entire range of the independent variable, in this case from $-\infty$ to ∞. The quantities μ and σ are known as the *mean* and *standard deviation* of the distribution. The intergral in (iv) may be obtained in any number of ways; an almost-too-clever way will be given shortly in these notes. For now, we might be content to look it up in a table.

The result obtained in (iv) is written as $\langle x^2\rangle=\sigma^2$. So, we have two funny notational things; N, which is a function of x, is written without an x, and $\langle x^2\rangle$, which is *not* a function of x, is written like one. Actually, what's happening is a special case of

$$\langle f\rangle=\int f(x)\,N\,dx,$$

which is a linear mapping, known as a *linear functional*, from the space of functions to the reals; If, f'rinstance, $f=x^2$, what we have is the image of the function f, which we choose to label $\langle x^2\rangle$; hence the terminology. You'll get used to it.

Anyhow, let's find the Fourier transform of $N(\mu, \sigma)$,

$$
\begin{aligned}
F(k) &= \int_{-\infty}^{\infty} N\, e^{ikx}\, dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\, e^{ikx}\, dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\, e^{ik(y+\mu)}\, dy \\
&= \frac{1}{\sigma\sqrt{2\pi}} e^{ik\mu} \int_{-\infty}^{\infty} e^{-\frac{1}{2}((y/\sigma)^2 - i2ky - (\sigma^2k^2))}\, e^{-\frac{1}{2}\sigma^2k^2}\, dy \\
&= \frac{1}{\sigma\sqrt{2\pi}} e^{ik\mu} e^{-\frac{1}{2}\sigma^2k^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y/\sigma - ik\sigma)^2}\, dy \\
&= \frac{1}{\sigma\sqrt{2\pi}} e^{ik\mu} e^{-\frac{1}{2}\sigma^2k^2} \int_{-\infty}^{\infty} e^{-z^2/2\sigma^2}\, dz \\
&= e^{ik\mu} e^{-\frac{1}{2}\sigma^2k^2}.
\end{aligned}
$$

Note that two changes of variables were made, $y = x - \mu$ and $z = y - ik\sigma^2$, and that the standard form for the last integral was used, as in the integral in (ii), above. One sneaky thing; in making the change of variable from y to z, the limits, strictly speaking, become $-\infty + ik\sigma^2$ to $\infty + ik\sigma^2$. This really is legitimate, although a detailed explanation and justification takes some time. What we are doing is changing our path of integration from the real axis to a contour in the complex plane that is parallel to the real axis, and as long as we don't cross any singularities of the integrand, we're okay. The integrand has no singularities except at $\pm i\infty$, so we don't come close to making the integral angry.

Also note a more general result: If $g(x) = f(x - \mu)$, and $G(k)$ and $F(k)$ are the Fourier transforms of $g(x)$ and $f(x)$, $G(k) = e^{ik\mu}\, F(k)$.

Anyhow, we have the result that the Fourier transform of a Gaussian is a Gaussian. We may also see why some of us prefer to define the transform with the factor of $\frac{1}{\sqrt{2\pi}}$, to preserve the symmetry. In addition, if

$$
\mathcal{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{N}(x)\, e^{ikx}\, dx, \qquad \int_{-\infty}^{\infty} \mathcal{F}(k)\mathcal{F}^*(k)\, dk = \int_{-\infty}^{\infty} \mathcal{N}(x)\mathcal{N}^*(x)\, dx.
$$

(This is shown in the notes **Normalization of Fourier Transforms**.)

Either way, we get $\sigma_k^2 = \langle k^2 \rangle = \frac{1}{\sigma^2}$ immediately, and the amazing result that $\sigma\sigma_k = 1$. It can be shown that for any function $f(x)$ and the transform $F(k)$, $\sigma\sigma_k \geq 1$, with equality holding only for a Gaussian, and that unless f is really pathological, the product $\sigma\sigma_k$ will never be very far from unity; it may be 6 or $\sqrt{2}\ln 2$, or something similar, but not 10^{23}.

Now, then, for any “reasonable” function f, consider

$$\frac{dF}{dk} = i \int_{-\infty}^{\infty} x\, e^{ikx}\, f(x)\, dx, \quad \ldots \quad \frac{d^n F}{dk^n} = (i)^n \int_{-\infty}^{\infty} x^n\, e^{ikx}\, f(x)\, dx,$$

so that

$$\int_{-\infty}^{\infty} x^n\, f(x)\, dx = (-i)^n\, \frac{d^n F}{dk^n}\Big|_{k=0}.$$

For this reason, $F(k)$ is sometimes known as the “Power Spectrum” of $f(x)$. To check for $N(0,\, \sigma)$, with $n = 2$,

$$F(k) = e^{-\sigma^2 k^2/2}, \quad \frac{d^2 F}{dk^2} = \sigma^2 e^{-\sigma^2 k^2/2} \left(\sigma^2 k^2 - 1\right),$$

so $(-i)^2 \frac{d^2 F}{dk^2}\big|_{k=0} = \sigma^2$, as before. This is similarly true for the inverse transform;

$$\int_{-\infty}^{\infty} k^n\, F(k)\, dk = \frac{(i)^n}{2\pi}\, \frac{d^2 f}{dx^2}\Big|_{x=0}.$$

This condition may give some insight into what we mean by “reasonable” $f(x)$ or $F(k)$; for instance, check out your answer to B&B 2.12(b) and see what you can say about $\int k^2\, F\, dk$ and $\frac{d^2 f}{dx^2}\big|_{x=0}$.

Okay, so what good are Fourier transforms? Lots! Here’s an example from physics; it ain’t 8.03, it’s a bit beyond 18.03, but it’s a real problem, a really good problem, and we can do it. Consider the *Heaty Quation*,

$$\nabla^2 T = D\, \frac{\partial T}{\partial t}. \tag{1}$$

Here, T is temperature and D is a constant,

$$D = \frac{(\text{mass density}) \times (\text{specific heat})}{(\text{thermal conductivity})}.$$

In one dimension, (1) becomes

$$\frac{\partial^2 T}{\partial x^2} = D\, \frac{\partial T}{\partial t}.$$

Our boundary conditions will be at $t = 0$; that is, we know $T(x,\, 0)$. Note that since (1) is only first-order in time ((1) is *not* the Wavy Quation), we need only one boundary condition. We will need to assume realistic conditions; as $x \to \pm\infty$, $T(x,\, 0) \to 0$, $\frac{\partial T}{\partial x}\big|_{t=0} \to 0$.

So, let

$$F(k,\, t) = \int_{-\infty}^{\infty} T(x,\, t)\, e^{ikx}\, dx,$$

and consider

$$\frac{\partial F}{\partial t} = \int_{-\infty}^{\infty} \frac{\partial T}{\partial t}\, e^{ikx}\, dx = \frac{1}{D} \int_{-\infty}^{\infty} \frac{\partial^2 T}{\partial x^2}\, e^{ikx}\, dx.$$

This last integral is done by parts;

$$\int_{-\infty}^{\infty} \frac{\partial^2 T}{\partial x^2}\, e^{ikx}\, dx = \frac{\partial T}{\partial x}\, e^{ikx}\Big|_{-\infty}^{\infty} - ik \int_{-\infty}^{\infty} \frac{\partial T}{\partial x}\, e^{ikx}\, dx = -ik \int_{-\infty}^{\infty} \frac{\partial T}{\partial x}\, e^{ikx}\, dx,$$

where the condition that $\frac{\partial T}{\partial x} \to 0$ as $x \to \pm\infty$ has been used. Similarly,

$$\int_{-\infty}^{\infty} \frac{\partial T}{\partial x}\, e^{ikx}\, dx = -ik \int_{-\infty}^{\infty} T\, e^{ikx}\, dx = -ikF(k,\, t).$$

Combining,

$$\frac{\partial F}{\partial t} = -\frac{k^2}{D}\, F \qquad (2).$$

This is readily solved; $F(k,\, t) = \mathcal{T}(k) e^{-k^2 t/D}$, where

$$\mathcal{T}(k) = F(k,\, 0) = \int_{-\infty}^{\infty} T(x,\, 0)\, e^{ikx}\, dx.$$

Thus, $\mathcal{T}$ is obtained from the initial conditions; we have determined $F(k,\, t)$, and hence can perform the inverse transform,

$$\begin{aligned} T(x,\, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k,\, t)\, e^{-ikx}\, dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} T(x',\, 0)\, e^{ikx'}\, dx' \right] e^{-k^2 t/D}\, e^{-ikx}\, dk. \end{aligned} \qquad (3)$$

Note the distinction between x and x'; $F(k,\, t)$ has no x-dependence. Now comes the fancy stuff; we switch the integrals!

$$T(x,\, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} T(x',\, 0) \left[\int_{-\infty}^{\infty} e^{-(k^2 t/D) - ik(x - x')}\, dk \right] dx' \qquad (4).$$

We can and will rewrite (4) as

$$T(x,\, t) = \int_{-\infty}^{\infty} T(x',\, 0)\, G(x,\, x',\, t)\, dx', \qquad (5)$$

where

$$G(x,\, x',\, t) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(k^2 t/D) - ik(x-x')}\, dk. \tag{6}$$

But, this is the inverse Fourier transform of $e^{-k^2 t/D}$, with $x - x'$ replacing x; we know that G is a Gaussian in $(x - x')$. In fact,

$$G = N\left(x',\, \sqrt{\frac{2t}{D}}\right) = \sqrt{\frac{D}{4\pi t}}\, e^{-(D/4t)(x-x')^2}. \tag{7}$$

If you wish, do the integral; it's similar to the integral done above, on page 2.

The function $G(x,\, x',\, t)$ is a *Green's Function* (hence the letter G). In general, if G comes from an equation that is linear, with constant coefficients (such as the Heaty Quation or the Wavy Quation), $G(x,\, x',\, t)$ will depend only on $x - x'$. You might have seen something similar in Green's functions for ODEs. The above form, equation (7), shows this explicitly.

In three dimensions, the integration by parts involves Green's theorem (*not* a coincidence on the names), and

$$G(\vec{x},\, \vec{x}',\, t) = \left(\frac{D}{4\pi t}\right)^{3/2} e^{-(D/4t)\left(\vec{x}-\vec{x}'\right)^2}.$$

Strictly speaking, what we have done is a "Fourier-Laplace" transform; Fourier in space, Laplace in time. In this case, the Laplace transform was really simple (see equation (2) and its solution).

Back to G: G is known as the "heat kernel". The term "kernel" comes from linear analysis; in operator notation,

$$\left(\frac{\partial^2}{\partial x^2} - D\frac{\partial}{\partial t}\right) G = 0,$$

and so $G \in$ the null space of the operator $\left(\frac{\partial^2}{\partial x^2} - D\frac{\partial}{\partial t}\right)$. Note that in this usage, D in *not* differentiation. From this, we can think of our approach to the problem as

a) Look for T such that $\frac{\partial^2 T}{\partial x^2} = -k^2 T$ and express T as a linear combination of such functions, or

b) Look for $\frac{\partial T}{\partial t} = -sT$ and look for a linear combination of these functions. Doing (a) and (b) simultaneously is the "Fourier-Laplace" transform.

$G(x,\, x',\, t)$ has the following properties:

i) $\int G(x,\, x',\, t)\, dx' = 1$

ii) If $x \neq x'$, $\lim_{t\to 0} G(x,\, x',\, t) = 0$

iii) If $x = x'$, $\lim\limits_{t\to 0} G(x,\, x',\, t) \to \infty$

Thus,

$$\lim_{t\to 0}\, G(x,\, x',\, t) = \delta(x' - x),$$

the infamous and illegal Dirac delta-"function". $\delta(x)$ has the property that

$$\int_{-\infty}^{\infty} f(x')\, \delta(x')\, dx' = f(0). \tag{8}$$

Check this in equation (5) above;

$$T(x,\, 0) = \lim_{t\to 0} \int T(x',\, 0)\, G(x,\, x',\, t)\, dx' = \int T(x',\, 0)\, \delta(x' - x)\, dx' = T(x,\, 0),$$

as it must.

We can generalize this; we have, from the Fourier transform and its inverse, $F(k)$ in terms of $f(x)$, and $f(x)$ in terms of $F(k)$, so we should be able to write $f(x)$ in terms of itself. Specifically,

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{-ikx}\, dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x')e^{ikx'}\, dx'\right]\, e^{-ikx}\, dk \\ &= \int_{-\infty}^{\infty} f(x') \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x'-x)}\, dk\right]\, dx'. \end{aligned}$$

Thus, the term in square brackets in the last line does what a delta-"function" does, and is a common representation for the delta-"function".

I've been putting "function" in quotes for a reason; the residual mathemetician in me recognizes that $\delta(x - x')$ is not a function at all. In fact, equation (8) is just a linear mapping from the space of functions into the reals; another linear functional. Since integrals of functions are a form of inner product, $\delta(x)$ does the same thing as a function in terms of its use as an inner product, but it can't be a function. The mapping $L(f) = f(0)$ is an *unbounded* linear functional, and $\delta(x)$ is sometimes known as a *distribution.* The term "distribution" was used to describe $N(\mu,\, \sigma)$, and they do similar things.

One last stinker; note that

$$\operatorname{sgn}(x) + 1 = 2\int_{-\infty}^{x} \delta(x')\, dx',$$

so that if $\operatorname{sgn}(x)$ were differentiable,

$$\frac{d}{dx}\operatorname{sgn}(x) = 2\,\delta(x).$$