Normalization of Fourier Transforms

In the notes on Fourier Transforms, an assertion was made to the effect that if

$$\mathcal{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx, \quad \text{then}$$

$$\int_{-\infty}^{\infty} \mathcal{F}(k)\mathcal{F}^*(k)\, dk = \int_{-\infty}^{\infty} f(x) f^*(x)\, dx.$$

Minor notational things; note the use of the complex cojnugate in both of the last two integrals. We know that in general, a Fourier transform of a real function is not necessarily real. The independent variables k, x and x' will be taken to be real. Also, the fancy "$\mathcal{F}$" is used to distinguish the above form of the Fourier transform from other forms, specifically B&B's form. Also note that, as in previous notes, $\sqrt{-1} = i$. Anyhow, we have

$$\mathcal{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx, \quad \mathcal{F}^*(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^*(x')\, e^{-ikx'}\, dx'.$$

Note how cleverly the second integral has been written in terms of x' instead of x. This is useful, because we can now see that

$$\begin{aligned} \mathcal{F}(k)\mathcal{F}^*(k) &= \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx \right] \left[\int_{-\infty}^{\infty} f^*(x')\, e^{-ikx'}\, dx' \right] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) f^*(x') e^{ik(x'-x)}\, dx\, dx', \end{aligned}$$

and so

$$\begin{aligned} \int_{-\infty}^{\infty} \mathcal{F}(k)\mathcal{F}^*(k)\, dk &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) f^*(x') e^{ik(x'-x)}\, dx\, dx'\, dk \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) f^*(x') \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x'-x)}\, dk \right] dx\, dx'. \end{aligned}$$

Here's the punchline; the term in square brackets is the return of the infamous and illegal δ-"function"! (Okay, I've slipped in $\delta(x) = \delta(-x)$; it's not hard to show.) We can now do either the x or x' integral to arrive at the result as given above,

$$\int_{-\infty}^{\infty} \mathcal{F}(k)\mathcal{F}^*(k)\, dk = \int_{-\infty}^{\infty} f(x) f^*(x)\, dx.$$

Note that this result does *not* hold if the factor in front of the Fourier transform is anything other than $\frac{1}{\sqrt{2\pi}}$. This factor is necessary becouse the equality of the integrals involves the *squares* of the functions. Very often, but far from always, it is this normalization, derived from the extension of an inner product to function spaces, that we want, and it's a good thing to have.