Partial Answers to B&B 2.12
with
even More Dirty Tricks

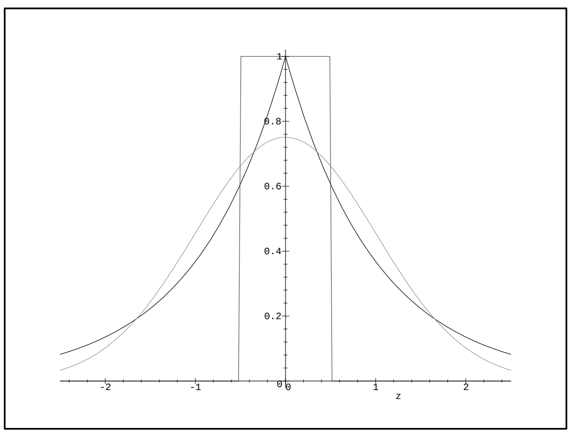
The Fourier transforms you should have got were:

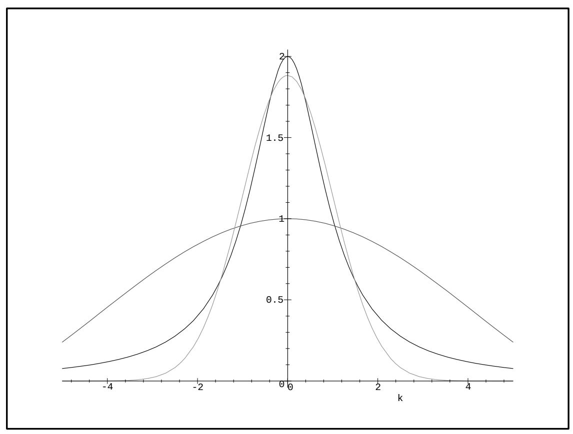
(i) $F_{\rm i} = \sqrt{2\pi/\alpha}\, e^{-k^2/2\alpha}$

(ii) $F_{\rm ii} = 2\beta(\beta^2 + k^2)^{-1}$

(iii) $F_{\rm iii} = (2/k)\sin(ka/2) = a(\sin u)/u$, $u = ka/2$.

Shown below are the plots of the functions f on the left and the Fourier transforms F on the right, with $\alpha = \beta = a = 1$. For plotting purposes only, the functions have been normalized so that $\int f^2\, dz = 1$, which means that $\int F^2\, dk = 2\pi$ for each (see the notes **Normalization of Fourier Transforms** for the reason why this is so). All calculations use the functional forms given in the text for $f_{\rm i}$, $f_{\rm ii}$ and $f_{\rm iii}$ and the above forms for $F_{\rm i}$, $F_{\rm ii}$ and $F_{\rm iii}$.

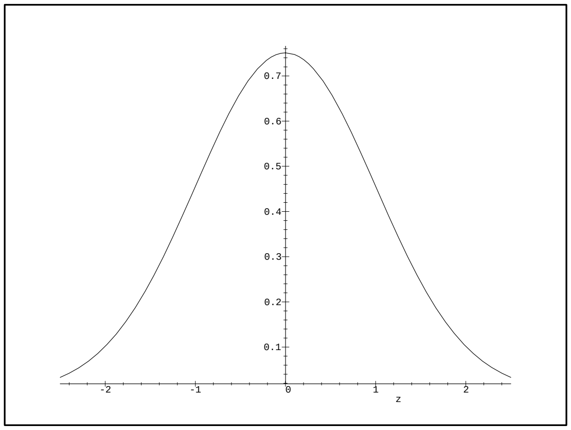


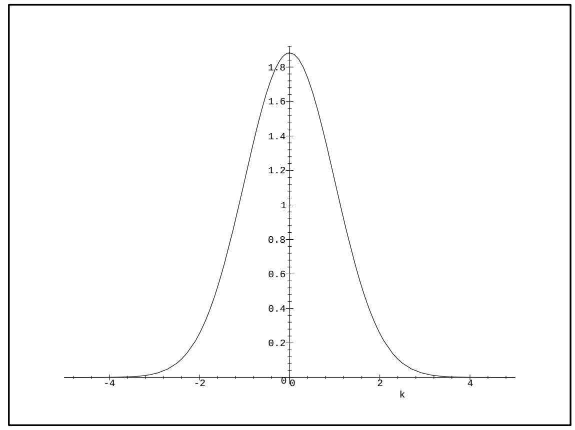


What I would like to do is investigate the widths of these functions, and how the widths of the functions $f_{\rm i}$, $f_{\rm ii}$ and $f_{\rm iii}$ are related to the widths of $F_{\rm i}$, $F_{\rm ii}$ and $F_{\rm iii}$, as suggested by the above figures. We will try to use the results obtained in the previous Supplemental Notes,

$$\langle z^2 \rangle = \frac{\int z^2\, f(z)\, dz}{\int f(z)\, dz} = -\frac{F''(0)}{F(0)}$$

$$\langle k^2 \rangle = \frac{\int k^2\, F(k)\, dk}{\int F(k)\, dk} = -\frac{f''(0)}{f(0)}.$$

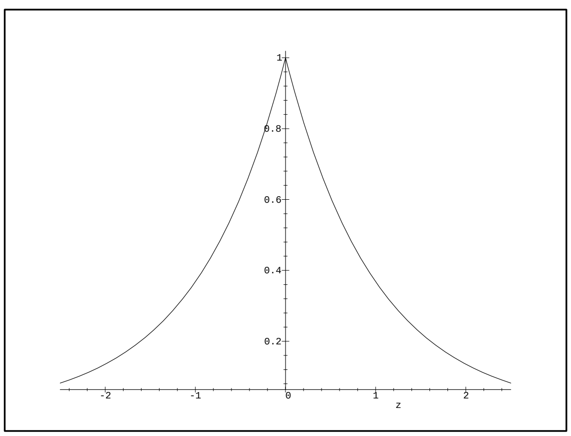


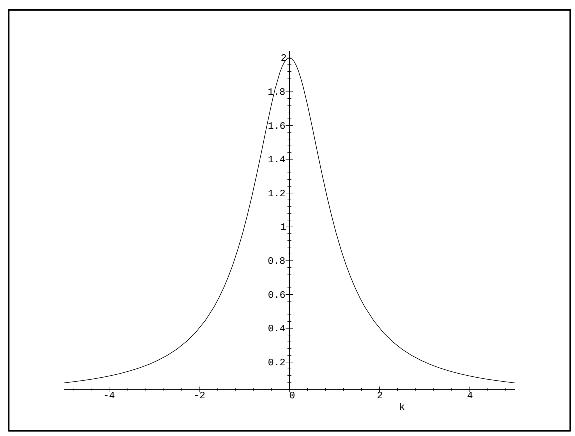


For (i): See $f_{\mathrm{i}}(z)$ on the left above and $F_{\mathrm{i}}(k)$ on the right. We have already seen that

$$F_{\mathrm{i}}''(k) = \sqrt{\frac{2\pi}{\alpha}}\, e^{-k^2/2\alpha} \left(-\frac{1}{\alpha} + \frac{k^2}{\alpha} \right),$$

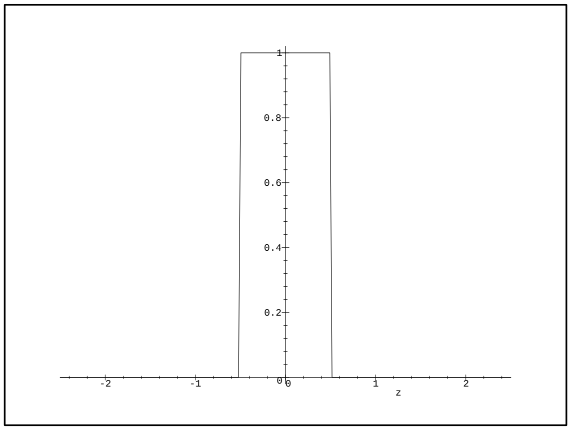
so $-F_{\mathrm{i}}''(0)/F_{\mathrm{i}}(0) = 1/\alpha$, and so the width of $f_{\mathrm{i}}(z)$ is $\sqrt{\langle z^2 \rangle} = \frac{1}{\sqrt{\alpha}}$. Since $F_{\mathrm{i}}(k)$ is also a Gaussian, the result is almost identical, and the width of $F_{\mathrm{i}}(k)$ is $\sqrt{\langle k^2 \rangle} = \sqrt{\alpha}$. These may be checked by direct integration.

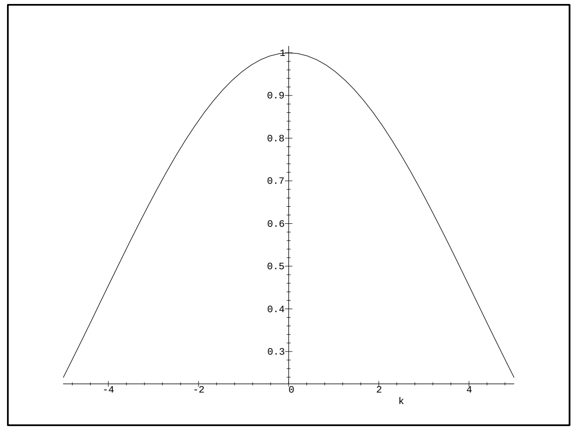




For (ii): See $f_{\mathrm{ii}}(z)$ on the left above and $F_{\mathrm{ii}}(k)$ on the right. By direct integration, $\langle z^2 \rangle = 2/\beta^2$ (do it, if you don't believe me, or even if you do; it's just the ratio of two factorials). You can check for yourself that $-F_{\mathrm{ii}}''(0) = 4/\beta^3$, $\langle z^2 \rangle = -F_{\mathrm{ii}}''(0)/F_{\mathrm{ii}}(0) = 2/\beta^2$, and so the width of f_{ii} is $\sqrt{\langle z^2 \rangle} = \sqrt{2}/\beta$. Finding the width of $F_{\mathrm{ii}}(k)$ is a situation where being too tricky will cost us. A casual glance at the form of $F_{\mathrm{ii}}(k)$ shows that $F(\beta/\sqrt{2}) = \frac{1}{2}F(0)$, so using $\beta/\sqrt{2}$, known as the "half-width at half maximum", gives us $\sqrt{\langle z^2 \rangle}(\mathrm{HWHM}) = 1$, and so we're done. The point of this exercise is that $f_{\mathrm{ii}}(z)$ is *not* what we would call a "reasonable function" for the purposes of using $F_{\mathrm{ii}}(k)$ as a power spectrum; note that f_{ii}' is discontinuous at $z = 0$, and $f_{\mathrm{ii}}''(0)$ does not exist. Another casual glance at

$\int k^2\, F_{\mathrm{ii}}(k)\, dk$ shows that the integral does not converge. If we wanted to be really hifalutin', we could say that this integral is infinty, as is the negative of the second derivative of f_{ii} at zero. This, however, will convince nobody.





For (iii): From the graph of $f_{\mathrm{iii}}(z)$ on the left above ($F_{\mathrm{iii}}(k)$ is on the right), any sane person would say that it has a half-width of $a/2$, and we will, too. Performing the integral gives $\langle z^2 \rangle = a^2/12$, for a width of $a/\sqrt{12}$. To find the derivatives of $F_{\mathrm{iii}}(k)$, it's perhaps best to use a Taylor's series expansion of F_{iii} in powers of u, $F \sim a(1 - u^2/6 + u^4/120)$, so $-F''(0)/F(0) = a^2/12 = \langle z^2 \rangle$. The width of F_{iii} is slightly problematical. First, note that $f_{\mathrm{iii}}^{(n)}(0) = 0$ for any n. But, the function F_{iii} has a readily identifiable width, in that it has a central peak and side lobes that decrease in size as $|k|$ increases. Thus, we could identify the width of F_{iii} as the half-width of the central peak, or where $u = ka/2 = \pi$, or $2\pi/a$. Then, we see that $\sqrt{\langle z^2 \rangle}(2\pi/a) = 2\pi/\sqrt{12}$, which is of order unity.

The above results are seen to be consistent with the first figure containing all of the plots. The functions $f_{\mathrm{i}}(z)$ and $f_{\mathrm{ii}}(z)$ are seen to have similar widths, while the width of $f_{\mathrm{iii}}(z)$ is noticeably smaller. In the graphs of the corresponding Fourier transforms, the widths of $F_{\mathrm{i}}(k)$ and $F_{\mathrm{ii}}(k)$ are similar, but $F_{\mathrm{ii}}(k)$ does not fall off as quickly with $|k|$ as does $F_{\mathrm{i}}(z)$, which we can take as reflecting the cusp of $f_{\mathrm{ii}}(z)$. The small width and rapid truncation of $f_{\mathrm{iii}}(z)$ is reflected in the very broad width of $F_{\mathrm{iii}}(k)$.