Larmor Radiated Power

(Note: for ease of typesetting and reading, for these notes vectors will be denoted by boldface instead of overarrows. F'rinstance, $\mathbf{v}$ instead of $\vec{v}$.)

B&B equation 4.13 can be written as

$$P_L = \frac{q^2\,a^2}{6\pi\epsilon_0 c^3} = \frac{q^2}{6\pi\epsilon_0 m c^3}\,ma^2 = \tau\,ma^2,$$

where $\tau = q^2/6\pi\epsilon_0 mc^3$ is a parameter, with dimensions of time, which may be calculated for any particle. F'rinstance, for an electron, proton and neutron,

$$\begin{aligned}\tau_{\mathrm{e}} &= 6.265 \times 10^{-24}\ \mathrm{s}\\ \tau_{\mathrm{p}} &= 3.411 \times 10^{-27}\ \mathrm{s}\\ \tau_{\mathrm{n}} &= 0 \qquad\qquad \text{(heh, heh)}.\end{aligned}$$

The Larmor power P_L is the rate at which an accelerated charge loses energy. P_L exists only when an acceleration exists, and only for charged particles. Suppose we model the radiation as being due to a radiative force $\mathbf{F}_{\mathrm{r}}$; $P_L = -\mathbf{F}_{\mathrm{r}} \cdot \mathbf{v}$, with the minus sign indicating that energy is *lost*.

We can find $\mathbf{F}_{\mathrm{r}}$ in terms of the energy lost in a given time;

$$\int P_L\,dt = -\int \mathbf{F}_{\mathrm{r}}\cdot\mathbf{v}\,dt = \int \tau m\,\dot{\mathbf{v}}\cdot\dot{\mathbf{v}}\,dt, \quad \text{so}$$

$$\begin{aligned}\int \mathbf{F}_{\mathrm{r}}\cdot\mathbf{v}\,dt &= -\tau m \int \dot{\mathbf{v}}\cdot\dot{\mathbf{v}}\,dt\\ &= -\tau m\,\dot{\mathbf{v}}\cdot\mathbf{v} + \tau m \int \mathbf{v}\cdot\ddot{\mathbf{v}}\,dt,\end{aligned}$$

where the last step is an integration by parts. Consider the first term on the right in the last expression; $\dot{\mathbf{v}} \cdot \mathbf{v} = \mathbf{a} \cdot \mathbf{v}$ is to be evaluated between the initial and final times. *If* we have i) $\mathbf{a} = \mathbf{0}$ initially and finally, *or* ii) $\mathbf{a} \perp \mathbf{v}$ (as in circular motion!), *or* iii) periodic motion, this term will vanish, and we have

$$\int \mathbf{F}_{\mathrm{r}}\cdot\mathbf{v}\,dt = \tau m \int \ddot{\mathbf{v}}\cdot\mathbf{v}\,dt.$$

Our identification of P_L with a radiative force then requires that $\mathbf{F}_{\mathrm{r}} = \tau m\ddot{\mathbf{v}}$.

(At this point, mathematicians might see where we'll run into trouble.)

Now, suppose we try to cause an acceleration by application of an external force $\mathbf{F}_{\rm e}$; then,

$$m\mathbf{a} = \mathbf{F} = \mathbf{F}_{\rm e} + \mathbf{F}_{\rm r} = \mathbf{F}_{\rm e} + \tau m\dot{\mathbf{a}}.$$

Rewrite this as $\mathbf{a} - \tau\dot{\mathbf{a}} = \mathbf{F}_{\rm e}/m$, and let $\mathbf{u} \equiv \mathbf{a}\, e^{-t/\tau}$ (an integrating factor), so

$$\begin{aligned}
\dot{\mathbf{u}} &= \dot{\mathbf{a}}\, e^{-t/\tau} - \frac{1}{\tau}\mathbf{a}\, e^{-t/\tau} \\
&= \frac{1}{\tau} e^{-t/\tau} \left[\tau\dot{\mathbf{a}} - \mathbf{a}\right] \\
&= \frac{1}{\tau} e^{-t/\tau} \left[-\frac{\mathbf{F}_{\rm e}}{m}\right].
\end{aligned}$$

Integrating (that's what integrating factors do),

$$\begin{aligned}
\mathbf{u}(t) &= -\frac{1}{m\tau} \int_{C_1}^{t} \mathbf{F}_{\rm e}(t')\, e^{-t'/\tau}\, dt', \\
\mathbf{a}(t) &= -\frac{1}{m\tau} \int_{C_1}^{t} \mathbf{F}_{\rm e}(t')\, e^{(t-t')/\tau}\, dt',
\end{aligned}$$

where C_1 is a constant of integration, with which we will deal later. Now, let $s = (t' - t)/\tau$, so $t' = t + \tau s$, $dt' = \tau\, ds$, and

$$\mathbf{a}(t) = -\frac{1}{m} \int_{C_2}^{0} \mathbf{F}_{\rm e}(t + \tau s)\, e^{-s}\, ds = \frac{1}{m} \int_{0}^{C_2} \mathbf{F}_{\rm e}(t + \tau s)\, e^{-s}\, ds.$$

If $\tau = 0$, $\mathbf{a} = \mathbf{F}_{\rm e}/m$, but

$$\mathbf{a}(t) = \frac{1}{m} \int_{0}^{C_2} \mathbf{F}_{\rm e}(t)\, e^{-s}\, ds = \frac{\mathbf{F}_{\rm e}}{m} \int_{0}^{C_2} e^{-s}\, ds.$$

The requirement that $\int_0^{C_2} e^{-s}\, ds = 1$ gives us $C_2 = \infty$, and so

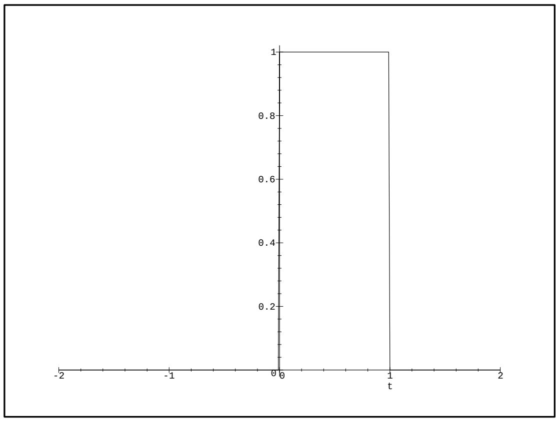
$$\mathbf{a}(t) = \frac{1}{m} \int_{0}^{\infty} \mathbf{F}_{\rm e}(t + \tau s)\, e^{-s}\, ds.$$

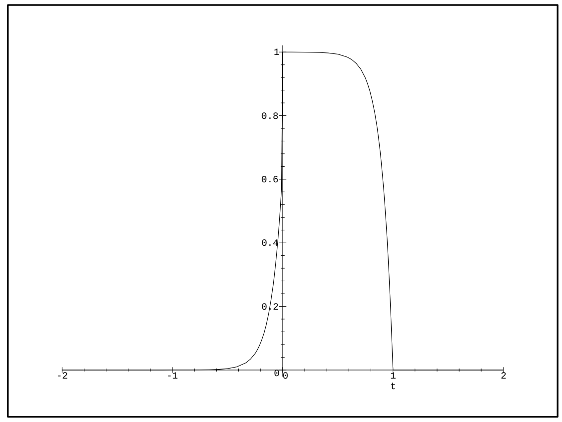
Well, so what? Well, we see that this relation would tell us that the acceleration at time t depends on the force at *later* times. Some of us are bothered by this.

For example: let $\mathbf{F}_\mathrm{e} = F\hat{x}$, $v_y = v_z = 0$. If

$$F(t) = \begin{cases} 0, & t < 0, \\ F_0, & 0 \le t \le T \\ 0, & t > T, \end{cases}$$

use of the above form for the acceleration gives $a_x(t)$ in the form shown. In the plots, the time t is given as a fraction of T, and $T = 10\tau$ has been used.





The acceleration begins before the force is applied, and the maximum acceleration, at $t = 0$, is $(F_0/m)\left[1 - e^{-T/\tau}\right]$.

Another example, which we will want later, is a sinusoidal force, such as $F = F_0 \cos\omega t$. In this case (you will do the integral sooner or later in an assignment),

$$a = \frac{F_0}{m} \frac{\cos(\omega t + \delta)}{\sqrt{1 + \omega^2 \tau^2}},$$

where $\delta = \tan^{-1} \omega\tau$. Again, a depends on F at future times, and $|a| < |F_0|/m$.

At this point, you may have to choose for yourself which part of physics has betrayed us. Mathematically, we have introduced a force which depends on acceleration, so our equations of motion are no longer merely second-order. One way of looking at the situation physically is to consider how we could apply the force "at time t" when we have an extended object. You might start by comparing the time τ for certain objects to the distance light can travel during that time (we will do this later).

B&B use the above result in a manner that hides the difficulty, but we're not afraid. For a more detailed discussion, including limitations on the validity of the above result, see Jackson's *Classical Electrodynamics*, pages 796-798.