## Special for 8.044, Spring 2006

We have already seen and used *Leibnitz's Formula* for differentiating an integral, in the form

$$\frac{d}{dy}\int_{x_L(y)}^{x_R(y)} f(x,\, y)\, dx = \frac{dx_R}{dy} f\left(x_R,\, y\right) - \frac{dx_L}{dy} f\left(x_L,\, y\right) + \int_{x_L(y)}^{x_R(y)} \frac{\partial}{\partial y} f(x,\, y)\, dx.$$

The above notation is similar to that used in the **18.023** text; the subscripts for $x_L(y)$ andd $x_R(y)$ are for "Left" and "Right," corresponding to the appearance of the graphs of these functions in the $x$-$y$-plane.

The formula can be extended in many ways; the limits could be functions of several variables, in which case partial derivatives would be used throughout. Or, for the case of multiple integrals to find probably distributions, the formula might be applied successively to nested integrals. Consider, for instance,

$$P(z) = \int_{x_L(z)}^{x_R(z)} dx \int_{y_B(x,\, z)}^{y_T(x,\, z)} f(x,\, y,\, z)\, dy.$$

Here, the subscripts on $y_B(x,\, z)$ and $y_T(x,\, z)$ are for "Bottom" and "Top," again corresponding to the appearance of the graphs of these functions in the $x$-$y$-plane (but different graphs for different values of the independent variable $z$).

Using Liebnitz's formula,

$$\begin{aligned}\frac{dP(z)}{dz} &= \frac{dx_R}{dz} \int_{y_B(x_R,\, z)}^{y_T(x_R,\, z)} f\left(x_R,\, y,\, z\right)\, dy - \frac{dx_L}{dz} \int_{y_B(x_L,\, z)}^{y_T(x_L,\, z)} f\left(x_L,\, y,\, z\right)\, dy \\ &\quad + \int_{x_L(z)}^{x_R(z)} dx\, \frac{\partial}{\partial z} \int_{y_B(x,\, z)}^{y_T(x,\, z)} f(x,\, y,\, z)\, dy.\end{aligned}$$

The partial derivative in the last term is now evaluated as

$$\begin{aligned}\frac{\partial}{\partial z} \int_{y_B(x,\, z)}^{y_T(x,\, z)} f(x,\, y,\, z)\, dy &= \frac{\partial y_T}{\partial z} f\left(x,\, y_T(x,\, z),\, z\right) - \frac{\partial y_B}{\partial z} f\left(x,\, y_B(x,\, z),\, z\right) \\ &\quad + \int_{y_B(x,\, z)}^{y_T(x,\, z)} \frac{\partial}{\partial z} f(x,\, y,\, z)\, dy.\end{aligned}$$

Whew. Altogether we have five terms:

$$\begin{aligned}
\frac{dP(z)}{dz} &= \frac{dx_R}{dz} \int_{y_B(x_R,\, z)}^{y_T(x_R,\, z)} f\left(x_R,\, y,\, z\right)\, dy \\
&- \frac{dx_L}{dz} \int_{y_B(x_L,\, z)}^{y_T(x_L,\, z)} f\left(x_L,\, y,\, z\right)\, dy \\
&+ \int_{x_L(z)}^{x_R(z)} \frac{\partial y_T}{\partial z} f\left(x,\, y_T(x,\, z),\, z\right)\, dx \\
&- \int_{x_L(z)}^{x_R(z)} \frac{\partial y_B}{\partial z} f\left(x,\, y_B(x,\, z),\, z\right)\, dx \\
&+ \int_{x_L(z)}^{x_R(z)} dx \int_{y_B(x,\, z)}^{y_T(x,\, z)} \frac{\partial}{\partial z} f(x,\, y,\, z)\, dy.
\end{aligned}$$

The above is the full-blown form, and those with masochistic streaks may continue the process to any number of nested integrals. However, for current (8.044) purposes, it's best not to get lost in the details.

That is, take another look at the orignal double integral,

$$P(z) = \int_{x_L(z)}^{x_R(z)} dx \int_{y_B(x,\, z)}^{y_T(x,\, z)} f(x,\, y,\, z)\, dy.$$

The independent variable $z$ appears five times in this expression, and so the derivative has five terms. We can summarize the process by saying: If $z$ appears in the limit of an integral, use the Fundamental Theorem of Calculus to differentiate the integral, multiplying by the derivative of the limit with the appropriate sign and evaluating the integrand at that limit. If $z$ appears in an integrand, take the partial derivative of the integrand with respect to $z$.

Consider for example the situation of Problem 3 from Problem Set 3. In this case, $P(z)$ ("$z$" is the area) can be written in a form such that $x_L$, $y_B$ and $f$ are constants, with the result that

$$\frac{d}{dz} P(z) = f \left[ \frac{dx_R}{dz} \int_{y_B}^{y_T(x_R,\, z)} dy + \int_{x_L}^{x_R(z)} \frac{\partial y_T}{\partial z}\, dx. \right].$$

So, a little bit of effort to show a useful technique results in a much easier calculation for those of us who still can't integrate by parts reliably.