Normal Modes for Continuous Systems

In class, we found solutions of the Wavy Quation

$$\psi = \psi(x,t), \quad \frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \tag{1}$$

for a stretched string of uniform mass per unit length $\mu = m_s/L$, under tension $\mathcal{T}$, so that $v^2 = \mathcal{T}/\mu$.

For the case of a string fixed at $x = 0$ but with a massive ring (mass M_r) free to slide perpendicular to the x-direction at $x = L$, the boundary conditions become

$$\psi(x,0) = 0, \quad M_r \frac{\partial^2\psi}{\partial t^2}\bigg|_{x=L} = -\mathcal{T}\frac{\partial\psi}{\partial x}\bigg|_{x=L}. \tag{2}$$

When solutions of the form

$$\psi(x,t) = X(x)\,T(t) = X(x)\,e^{i\omega t}$$

are sought (note the distinction between the tension $\mathcal{T}$ and the function T), (1) and (2) become, after using separation of variables and factoring out $T(t)$,

$$X'' + \frac{\omega^2}{v^2}\,X = 0, \quad X(0) = 0, \quad \frac{\omega^2}{v^2}\,X(L) = \frac{\mu}{M_r}\,X'(L). \tag{3}$$

We can find explicit solutions to (3) in terms of the roots of a transcendental equation, and you have a problem investigating the orthogonality of those solutions by direct integration. These notes will show how the proper orthogonality relation may be demonstrated without finding explicit forms for $X(x)$.

Let $X_m(x)$ be a solution of (3) with $\omega = \omega_m$, and X_n be a solution with $\omega = \omega_n$. Then, consider the combination

$$X''_m\,X_n - X_m\,X''_n = \frac{d}{dx}\left[X'_m\,X_n - X_m\,X'_n\right].$$

(Fill in the missing step!) But, from (3), $X''_m = -\frac{\omega_m^2}{v^2}\,X_m$, $X''_n = -\frac{\omega_n^2}{v^2}\,X_n$, so

$$\left(\omega_n^2 - \omega_m^2\right)\,X_m\,X_n = \frac{d}{dx}\left[X'_m\,X_n - X_m\,X'_n\right].$$

Integrating from 0 to L,

$$\begin{aligned}\frac{1}{v^2}\left(\omega_n^2 - \omega_m^2\right)\int_0^L X_m(x)\,X_n(x)\,dx &= \left[X'_m(x)\,X_n(x) - X_m(x)\,X'_n(x)\right]_0^L \\ &= X'_m(L)\,X_n(L) - X_m(L)\,X'_n(L).\end{aligned}$$

Using the boundary condition at L (from (3)) to eliminate X'_m and X'_n in favor of X_m and X_n,

$$\frac{1}{v^2}\left(\omega_n^2 - \omega_m^2\right)\int_0^L X_m(x)\,X_n(x)\,dx = \frac{M_r}{\mu v^2}\left(\omega_m^2 - \omega_n^2\right) X_m(L)\,X_n(L).$$

So, if $\omega_m^2 \neq \omega_n^2$,

$$\int_0^L \mu\,X_m(x)\,X_n(x)\,dx + M_r X_m(L)\,X_n(L) = 0,$$

and this is our orthogonality relation. The important physical point to note is that the integral represents "breaking the string into small pieces", multiplying by the mass ($\mu\,dx$) of each piece, multiplying by the displacements of the independent modes at that point, and adding. The term $M_r\,X_m(L)X_n(L)$ corresponds to the finite mass at $x = L$.

The above is a specific application of the theory of *Sturm-Liouville* problems. For more on this subject, consult any decent Differential Equations text or ESG's self-paced study units and notes, specifically **Why we care about Bessel Functions**, linked from the page at

`http://web.mit.edu/18.03-esg/www/notes/TofC.html`