Plasma Oscillations

As mentioned in lecture today (Friday, April 25), the dispersion relation for electromagnetic waves in a plasma may be obtained either by regarding the displacement of the electrons as contributing to the polarization vector $\vec{\boldsymbol{P}}$ or by regarding the motion of the electrons as part of a conduction current. These notes show that treating the electrons as free electrons subject to the oscillating electric field and the resulting motion as a conduction current yields the same dispersion relation as obtained in the given solutions to Problem 48.

For these notes, the relative permeability will be taken to be unity ($\kappa_{\mathrm{M}} = 1$) and the motion of the electrons will be modeled as a conduction current, so that κ_{E} will be taken to be 1. The number of electrons per unit volume will be N (some texts use n, but that symbol is overused already). Vectors will be boldface italic with an overarrow (or hat for units vectors).

The current density $\vec{\boldsymbol{J}}$ is in general then

$$\vec{\boldsymbol{J}} = Nq\vec{\boldsymbol{v}} = -Ne\vec{\boldsymbol{v}},$$

where $q = -e$ for electrons has been used. The motion of the electrons, for $\vec{\boldsymbol{v}}$ uniform, does not change the overall electrical neutrality of the plasma, so the charge density is $\rho = 0$. (Note that this is *not* the case for the optional part (g) of Problem 48.) Equations (6.27) on Page 413 then reduce to a form similar to Equations (6.36) on Page 421, summarized here:

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}} = 0 \tag{a}$$

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} = -\frac{\partial\vec{\boldsymbol{B}}}{\partial t} \tag{b}$$

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} = \mu_0\epsilon_0\frac{\partial\vec{\boldsymbol{E}}}{\partial t} - \mu_0 Ne\vec{\boldsymbol{v}} \tag{c}$$

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}} = 0. \tag{d}$$

There are many ways to proceed from here. For the purposes of finding the dispersion relation, the most direct is to take the curl of both sides of (b), using the fact that the divergence of $\vec{\boldsymbol{E}}$ vanishes, to obtain

$$\vec{\boldsymbol{\nabla}}\times\left(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}}\right) = -\boldsymbol{\nabla}^2\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\times\left(\frac{\partial\vec{\boldsymbol{B}}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}\right),$$

where the equality of mixed partials has been used without apology. Taking the time derivative of (c) and substituting the above, and using $\frac{d}{dt}\vec{\boldsymbol{v}} = \vec{\boldsymbol{a}}$, the electron acceleration, yields

$$\boldsymbol{\nabla}^2 \vec{\boldsymbol{E}} = \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{\boldsymbol{E}} - \mu_0 N e \vec{\boldsymbol{a}}. \qquad \clubsuit$$

At this point, it's easiest to use the form for the electric field given in Problem 48,

$$\vec{\boldsymbol{E}} = E_0 \cos(\omega t - kz) \hat{\boldsymbol{x}}.$$

Finding the vector Laplacian is straightforward, and $\boldsymbol{\nabla}^2 \vec{\boldsymbol{E}} = -k^2 \vec{\boldsymbol{E}}$. Similarly, $\frac{\partial^2}{\partial t^2} \vec{\boldsymbol{E}} = -\omega^2 \vec{\boldsymbol{E}}$. The acceleration $\vec{\boldsymbol{a}}$ is, from Newton's Second Law, $\vec{\boldsymbol{a}} = (-e/m) \vec{\boldsymbol{E}}$. Combining these in $\clubsuit$ and using $\mu_0 \epsilon_0 = 1/c^2$ gives

$$(-k^2 c^2) \vec{\boldsymbol{E}} = (-\omega^2) \vec{\boldsymbol{E}} + \left(\frac{N e^2}{m \epsilon_0} \right) \vec{\boldsymbol{E}},$$

which can only be valid if

$$c^2 k^2 = \omega^2 - \frac{N e^2}{m \epsilon_0} = \omega^2 - \omega_{\mathrm{p}}^2,$$

the same dispersion relation obtained previously.

The same result may be obtained from Equation (6.42) if the conductivity σ is properly interpreted. With $\kappa_{\mathrm{E}} = \kappa_{\mathrm{M}} = 1$, that relation becomes

$$k^2 = \frac{\omega^2}{c^2} - j \omega \mu_0 \sigma$$

and was derived for an electric field of the form given in Equation (6.38), which is, with a minor change in notation,

$$\vec{\boldsymbol{E}} = E_0 e^{(j \omega t - kz)} \hat{\boldsymbol{x}}.$$

Use of Equation (6.42) necessitates a relation between $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{J}}$ and hence between $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{v}}$. Finding the acceleration as above and integrating,

$$\vec{\boldsymbol{a}} = \left(\frac{-e}{m} \right) \vec{\boldsymbol{E}}, \qquad \vec{\boldsymbol{v}} = \left(\frac{-e}{j \omega m} \right) \vec{\boldsymbol{E}}.$$

This gives a conductivity that is purely imaginary, $\sigma = -j N e^2 / \omega m$, and substitution into Equation (6.42) while counting minus signs very carefully gives

$$k^2 = \frac{\omega^2}{c^2} - \mu_0 \left(\frac{N e^2}{m} \right) = \frac{\omega^2}{c^2} - \frac{1}{c^2} \left(\frac{N e^2}{\epsilon_0 m} \right) = (1/c^2) \left(\omega^2 - \omega_{\mathrm{p}}^2 \right).$$

Two things (at least) should be noted in the above use of the conductivity σ: First, it may seem that the direction of propagation of the wave might matter. This is not the case; if $\vec{\boldsymbol{E}} = E_0 e^{j(-\omega t - kz)} \hat{\boldsymbol{x}}$ were used, for instance, instead of Equation (6.38), then the conduction term in Equation (6.42) would be $+j\omega\mu_0\kappa_{\mathrm{M}}\sigma$, with the plus sign instead of the minus. When integrating the expression for the acceleration to find the velocity, $\vec{\boldsymbol{v}} = \vec{\boldsymbol{a}}/(-j\omega)$, and the minus sign is recovered.

The above is a special case of that considered in the text leading up to Equation (6.71) on Page 445. Note that in that equation, the sign of the charge does not matter; which we call “positive” or “negative” should not matter in this case, and it doesn’t.