Math Tricks needed for the Planck Formula

Unlike other notes, these aren't really supplemental; their purpose is to fill in many gaps in B&B's derivations in Section 4.5. We will be going over the physics in class, and rather than spending time doing what is essentially math (and having you try to take voluminous notes instead of appreciating what's going on), I'd like you to have the tricks in front of you. I'll be quoting results, and you can refer to these notes for derivations.

Reading and understanding the math herein will make you a better person. If there are some things you don't see or appreciate right away, by all means ask.

From kinetic theory, we know that the probability that an entity (particle, mode, degree of freedom) has an energy α when in equilibrium with its surroundings at Kelvin temperature T is proportional to $e^{-\alpha\beta}$, where $\beta = 1/kT$. We are interested in finding expectation energies, and we will see that a continuum of allowed energies gives a different form for the expectation energy than a system that allows only discrete energies.

Classically, when all energies above a certain minimum are allowed, the expectation energy is

$$\langle E \rangle = \frac{\int_0^\infty \alpha\, e^{-\alpha\beta}\, d\alpha}{\int_0^\infty e^{-\alpha\beta}\, d\alpha} = \frac{1}{\beta} = kT$$

(Do the integrals yourself - that's the point of these notes.). This is no large deal. But, what if the lowest possible energy is not 0, as in the above integral, but some lowest energy E_0? Then, the above form for $\langle E \rangle$ becomes

$$\langle E \rangle = \frac{\int_{E_0}^\infty \alpha\, e^{-\alpha\beta}\, d\alpha}{\int_{E_0}^\infty e^{-\alpha\beta}\, d\alpha} = E_0 + \frac{1}{\beta} = E_0 + kT$$

(again, you do the integral). Thus, the expected energy is kT above the lowest energy, and if we set this lowest energy to zero (allowable classically), then $\langle E \rangle = kT$, as before.

Although it may seem unnecessary at this point, let's introduce a useful trick. Let

$$N(\beta) = \int_0^\infty e^{-\alpha\beta}\, d\alpha.$$

Here, “N” is for “Normalization”, not “Number”; please do not confuse it with B&B’s $N(\omega)$ in Equations 4.64 and 4.65. Then, we have

$$\int_0^\infty \alpha\, e^{-\alpha\beta}\, d\alpha = -\frac{dN}{d\beta},$$

and

$$\langle E\rangle = -\frac{dN/d\beta}{N} = -\frac{d}{d\beta}\ln N.$$

Now, suppose we know the standard tricks of integration, but we either don’t know or are too lazy to look up the integral of an exponential. That is, we know that by making a change of variables $u = \alpha\beta$ we have $N(\beta) = \frac{1}{\beta}\int e^{-u}\, du$. Then, $\ln N = -\ln\beta + \ln(\int e^{-u}\, du)$. But, the second term in the sum does not depend on β at all; in fact, the integral is 1, and it’s well-known that $\frac{d1}{d\beta} = 0$. Our result is that

$$\langle E\rangle = -\frac{d}{d\beta}(-\ln\beta) = \frac{1}{\beta},$$

as before. As an extra motivation, note that if

$$N_{E_0}(\beta) = \int_{E_0}^\infty e^{-\alpha\beta}\, d\alpha = e^{-E_0\beta} N(\beta),$$

$$\frac{d}{d\beta}\left(-\ln N_{E_0}(\beta)\right) = E_0 + \frac{1}{\beta}$$

follows readily. (The above “trick” is similar to that used to find moments of distributions in the notes on **Fourier Transforms** including **More Dirty Tricks**.)

Now, suppose we have a discrete system, and that the allowed energies are $E_n = n\,\Delta E$, where ΔE is the constant energy-level spacing. Note that in this form, the lowest energy corresponds to the lowest allowed value of n. As we have seen, we are free to set this lowest energy, and hence the lowest value of n, equal to 0. Then, we have

$$N(\beta) = \sum_{n=0}^{\infty} e^{-n\Delta E\beta} = \frac{1}{1 - e^{-\Delta E\beta}},$$

where the usual form for a geometric series has been employed to do the sum. Notice a couple of things; first, for the sum to be finite, $\beta > 0$, and we can interpret this as meaning that there is no such thing as a negative temperature (actually, this is not a really good argument, but it’s worth noting in passing). Second, unlike the continuum case, $N(\beta)$ depends on ΔE; there was no corresponding parameter in the continuum case.

Well, now we're all set. A straightforward calculation (DO IT! I'm about to skip some steps) gives

$$\langle E \rangle = \frac{\Delta E}{e^{\Delta E \beta} - 1}.$$

Substituting $\Delta E = h\nu = \hbar\omega$ and $\beta = \frac{1}{kT}$ gives B&B Equation 4.67.

Since I can't depend on anyone's successfully taking up the Toscanini's challenge, I'll show how to get the result for the total energy density integrated over frequency, $U(T)$, as done in part (b) on page 312. This is not just showing off; such integrals are common in statistical mechanics, and similar integrals are done whenever you have a weighted average, with the weighting function similar to Equation 4.68.

Starting from equation 4.68, but using $\hbar$ instead of h,

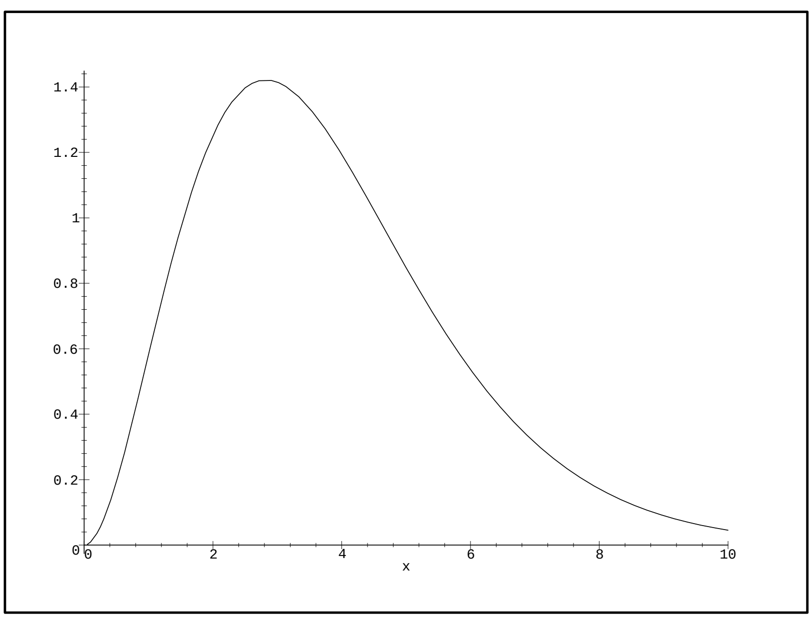
$$U(\omega, T)\, d\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3\, d\omega}{e^{\hbar\omega/kT} - 1}.$$

We wish to find

$$\begin{aligned} U(T) = \int_0^\infty U(\omega, t)\, d\omega &= \frac{\hbar}{\pi^2 c^3} \int_0^\infty \frac{\omega^3\, d\omega}{e^{\hbar\omega/kT} - 1} \\ &= \frac{\hbar}{\pi^2 c^3} \left(\frac{kT^4}{\hbar} \right) \int_0^\infty \frac{x^3\, dx}{e^x - 1}, \end{aligned}$$

where the same substitution as in B&B, $x = \hbar\omega/kT$, has been used.

Note that all of the physical constants appear to the left of the integral; the integral is a pure number. We can "guess" what it is, aided by the plot below.



From the plot, and as stated on Page 312, the integrand peaks at $x < 3$; between 0 and 3, the integrand is no greater than x^2, so $\int_0^3 x^2\,dx = 9$ is a decent upper bound. Here's how to do it exactly:

$$\frac{x^3}{e^x - 1} = \frac{x^3\,e^{-x}}{1 - e^{-x}} = x^3\,e^{-x}\sum_{n=0}^{\infty}\left(e^{-x}\right)^n = x^3\sum_{n=1}^{\infty} e^{-nx}$$

(Check this this!). So,

$$\begin{aligned}
\int_0^\infty \frac{x^3}{e^x - 1}\,dx &= \int_0^\infty x^3\sum_{n=1}^{\infty} e^{-nx}\,dx \\
&= \sum_{n=1}^{\infty}\int_0^\infty x^3\,e^{-nx}\,dx \\
&= \sum_{n=1}^{\infty}\int_0^\infty y^3\,e^{-y}\,dy\left(\frac{1}{n^4}\right),
\end{aligned}$$

where the substitution $y = nx$ has been used. The integral is $\int y^3\,e^{-y}\,dy = 3! = 6$ (look it up or do it!), so

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = 6\sum_{n=1}^{\infty}\frac{1}{n^4} = 6\,\frac{\pi^4}{90} = \frac{\pi^4}{15},$$

where the last sum is one that we found from Fourier series way back in Chapter 2 (I *told* you we would need that one).

Having done this, it is only fair to point out that such an improper definite integral can be done my MAPLE without any difficulty.

Our final result is

$$U(T) = \frac{\pi^2}{15}\frac{k^4 T^4}{\hbar^3 c^3}$$

(compare to B&B page 312). As is explained in the notes on **Blackbody Radiation**, the more common quantity is the flux

$$S(T) = \frac{c}{4}\,U(T) = \sigma\,T^4.$$