Pulses on a Struck String

These notes investigate specific examples of transverse motion on a stretched string in cases where the string is at some time undisplaced, but with a non-zero transverse velocity.

Consider the *Cauchy Problem*

$$u_{tt} = c^2\, u_{xx}, \qquad u(x,\, 0) = f(x) \qquad u_t(x,\, 0) = g(x), \tag{1}$$

where the subscripts denote partial differentiation.

What follows might well annoy mathematicians. For a more rigorous treatment, including of course the needed differentiability and smoothness of the functions, see *Differential Equations: A Modeling Approach*, Robert L. Borrelli and Courtney S. Coleman (Prentice-Hall, 1987), Section 13.2 (hereinafter referred to as "B&C"; a newer edition has been published).

The solution to Problem (1) is shown in B&C to be

$$u(x,\, t) = \frac{1}{2}\left[f(x - c\,t) + f(x - c\,t)\right] + \frac{1}{2c}\int_{x-c\,t}^{x+c\,t} g(s)\, ds. \tag{2}$$

The remainder of these notes will consider the case $f(x) \equiv 0$, so that Equation (2) simplifies to

$$u(x,\, t) = \frac{1}{2c}\int_{x-c\,t}^{x+c\,t} g(s)\, ds. \tag{3}$$

Note that in the limit of small $\Delta t > 0$, this reduces to $u \approx g(x)\, \Delta t$.

Equation (3) is often written in the form

$$u(x,\, t) = P(x + c\,t) + Q(x - c\,t) \tag{4}$$

where $\qquad P(s) = \dfrac{1}{2c}\displaystyle\int_0^s g\left(s'\right)\, ds', \qquad Q(s) = -\dfrac{1}{2c}\displaystyle\int_0^s g\left(s'\right)\, ds'.$

It is common to interpret Equation (4) in terms of "pulses" traveling in the negative- and positive-x directions respectively. Part of the motivation for these notes is to show that this mathematical interpretation may lead to a physical interpretation that is somewhat counterintuitive. For now, consider that in both $P(s)$ and $Q(s)$, the lower limit of zero is arbitrary, and changing this lower limit corresponds to a constant of integration added to one "pulse" and subtracted from the other.

Before looking at examples, note that the problem in (1) above is *not* a boundary-value problem. Restricting x, and hence the range of both $f(x)$ and $g(x)$ to a finite interval, would require suitable periodic extensions of $f(x)$ and $g(x)$; these periodic extensions will not be discussed in detail here, although they are clearly implied in any use of Fourier Series.

Although it may not be needed for mathematical analysis, for physical interpretation it might be helpful to recall kinematic and dynamic properties of $u(x, t)$ and its derivatives. For small displacements on a uniform string of mass density μ and subject to a tension $\mathcal{T}$, we have:

- **Speed of Propagation** (denoted c, as above):

$$c^2 = \frac{\mathcal{T}}{\mu}.$$

- **Potential Energy Density** (denoted U):

$$U = \frac{1}{2}\,\mathcal{T}\,u_x^2.$$

- **Kinetic Energy Density** (denoted K):

$$K = \frac{1}{2}\,\mu\,u_t^2.$$

- **Total Energy Density**

$$U + K = \frac{1}{2}\left[\mathcal{T}\,u_x^2 + \mu\,u_t^2\right] = \frac{\mu}{2}\left(c^2\,u_x^2 + u_t^2\right).$$

- **Longitudinal Power Density** (denoted p):

$$p = -\mathcal{T}\,u_x\,u_t.$$

- **Conservation of Energy**:

$$\frac{\partial}{\partial t}\,(U + K) + \frac{\partial p}{\partial x} = 0.$$

- **Transverse Momentum Density** (denoted $\mathbf{p}_\mathrm{T}$):

$$\mathbf{p}_\mathrm{T} = \hat{\mathbf{u}}\,u_t.$$

Five examples will be considered in what follows. Some of the above quantities may be infinite. We won't worry. Much.

The five examples will be:

- A standing wave on a string.
- A unit (δ-function) impulse on an unbounded string (no reflection).
- A square-wave impulse on an unbounded string.
- A smoother impulse on an unbounded string.
- The same impulse on a bounded string.

Standing Wave on a String

As will be seen, the standing wave, being spatially periodic, may be a solution for an unbounded string or a bounded string with linear boundary conditions. The example is taken from the **18.023** text, *Calculus: An Introduction to Applied Mathematics*, H. P. Greenspan and D. J. Benney (Breukelen, 1997), Page 505. In the current notation, we have

$$f(x) \equiv 0, \qquad g(x) = \cos x.$$

A basic use of Equation (3) above gives

$$
\begin{aligned}
u(x,\, t) &= \frac{1}{2c} \left[\sin(x + c\,t) - \sin(x - c\,t) \right] \\
&= \frac{1}{c} \cos x \sin ct.
\end{aligned}
$$

This standard result shows that a standing wave may be represented as the sum of two traveling waves, and vice versa.

A δ-function Impulse

At the risk of offending most mathematicians and some physicists, let

$$f(x) \equiv 0, \qquad g(x) = A\,\delta(x).$$

All we'll really need is that the δ-function is the "derivative" of the heaviside function $H(x)$, where

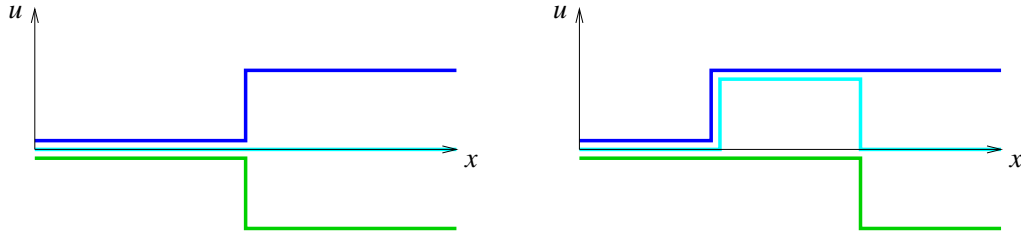$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases}$$

If we made an attempt to be rigorous, we would want $H(0) = 1/2$, but that's not our goal, so the matter won't be mentioned in these notes.

The result of using Equation (3) is then

$$u(x,\, t) = \frac{A}{2\,c} \left[H(x + c\,t) - H(x - c\,t) \right],$$

clearly a "pulse" of height $A/2c$ expanding in both the positive- and negative-x directions with speed c. Unfortunately, the energy density, while localized, is infinite (as is its integral; δ-functions are like that). Momentum is finite and conserved.

Consider the expression in Equation (4); as defined, both $P(s)$ and $Q(s)$ are zero for $s < 0$ and $P(s) > 0$ for $s > 0$, $Q(s) < 0$ for $s > 0$ (for $A > 0$). This gives, as expected, $u(x, t) = 0$ for $|x| > ct$, but for $x < ct$, this zero net wave is the sum of two nonzero waves in the region beyond where the "signal" could have propagated. This is (sort of) mathematically correct, but the physical interpretation is perhaps counterintuitive. The pulses are represented in the figure below, the pulse corresponding to $P(x + ct) \geq 0$ in blue, moving to the left in the figure, and $Q(x - ct) \leq 0$ in green moving to the right, with the sum in cyan (the blue and green pulses are displaced for clarity).

A certain symmetry might be restored by adding $1/2$ to $Q(s)$ and subtracting $1/2$ from $P(s)$, but this still has the "traveling waves" $P(x + ct)$ and $Q(x - ct)$ nonzero in regions beyond the impulse at $t = 0$. Any boundary conditions would not affect this result.

It should be noted that the factor "A" introduced must have dimensions of length^2/time; A may be thought of as the u-component of the imparted momentum divided by μ. For further physics interpretations, whenever any displacement of a wave traveling on a stretched string is shown with a sharp edge, the transverse velocity profile will include a δ-function part; if such functions are not desirable, then such waves should not be used.
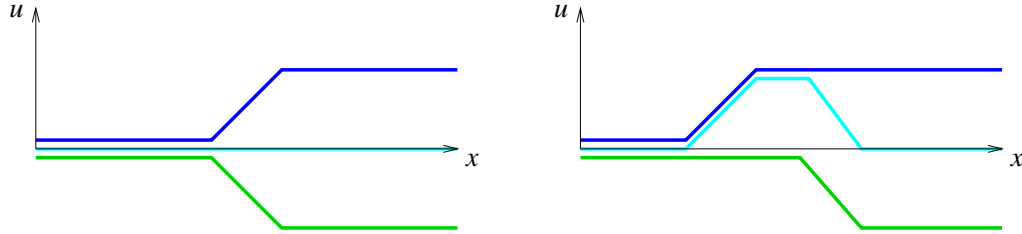
Square-Wave Impulse

Represent the impulse in terms of Heaviside functions, so that

$$f(x) \equiv 0, \qquad g(x) = B \left[H \left(x + x_0 \right) - H \left(x - x_0 \right) \right].$$

In words, $g = 0$ for $|x - x_0| > 0$, $g = B$ for $|x - x_0| < 0$.

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The solution to Problem (1) may then be found by various means. In the animation which accompanies these notes, the same program that generated the animation calculated the integral in Equation (3) without complaint. Calculation "by hand" is often left as an exercise (as in B&C, Page 520, Problem 1). Presented below is a graphical interpretation.

As can be seen, as the time inreases, the "pulse" will spread, but the energy and momentum will remain constant. In this example, it's clearer that the pulse is "smoother" than the impulse. Mathematically, this is represented by the integral in D'Alembert's solution (Equation (2)). Physically, this if often interpreted by recognizing that at some point x with $|x - x_0| > 0$, the impulse imparted to different parts of the string take different times to be "seen". (A more detailed discussion, with acknowledgement of physicists' sensibilities, is in B&C, Page 519.)

The temptation to continue with other polynomials for $g(x)$ restricted to a finite interval will be only partially resisted. Analytic calculations will, for now, remain as exercises. However, the case of an impulse that is a parabolic function of x will be considered *via* computer-generated animations.

Parabolic Impulse

The initial condition used to generate the impulse was

$$f(x) \equiv 0, \qquad g(x) = [H(x+1) - H(x-1)] \left(1 - x^2\right)$$

(for the purposes of generating the animiation, all parameters and dimensions are set to unity). The animation shows both the smoothing and spreading of the pulse.

Parabolic Impulse - Bounded String

The previous four examples did not consider boundary conditions, basically assuming an infinite string, or a finite string but for times sufficiently short that any reflections from boundaries are not considered. To see the effect of reflection,
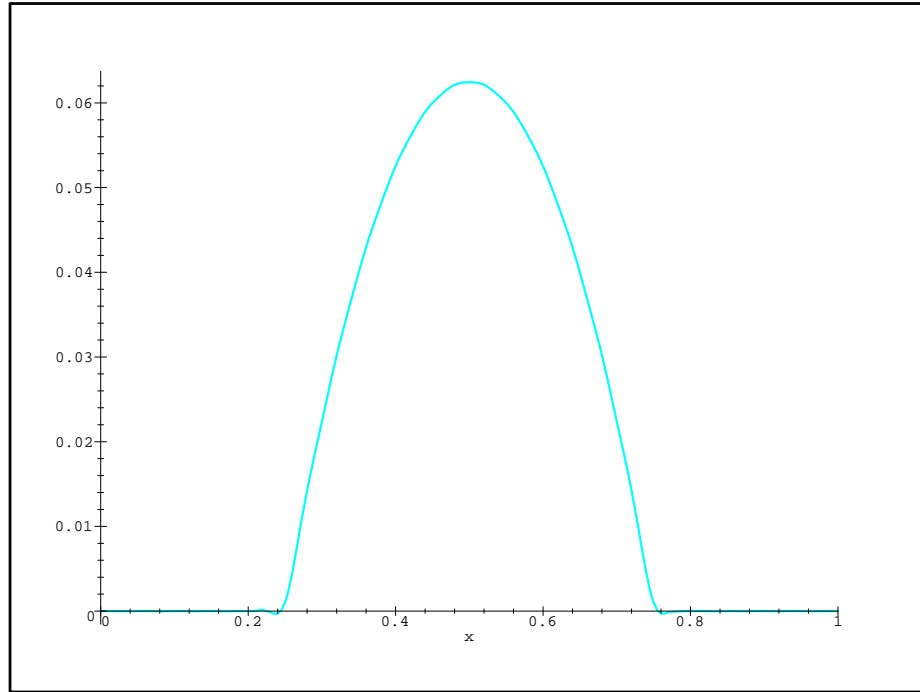
this example assumes that pulses are inverted upon reflection, so that a Fourier sine series may be used.

The Cauchy Problem is now a boundary-value problem. For animation purposes, the initial data are

$$f(x) \equiv 0 \qquad g(x) = [H\,(x - 1/4) - H\,(x - 3/4)]\,(s - 1/4)(3/4 - s)$$

$$u(0,\, t) = u(1,\, t) \equiv 0.$$

The Fourier coefficients C_n of $g(x)$ were found by computer, and as long as a computer was being used, the first fifty terms of the series were summed, with the result shown here. Note that at this resolution, the Fourier sum is barely distinguishable from the parabola.

The animation depicts

$$\sum_{n=1}^{50} \frac{C_n}{n\,\pi}\,\sin(n\,\pi\,x)\,\sin(n\,\pi\,t)$$

and is "looped" at the fundamental period of $T = 2$. The animation shows both the spreading of the pulse and the larger slopes ("unsmoothing") at the boundaries due to inversion upon reflection.