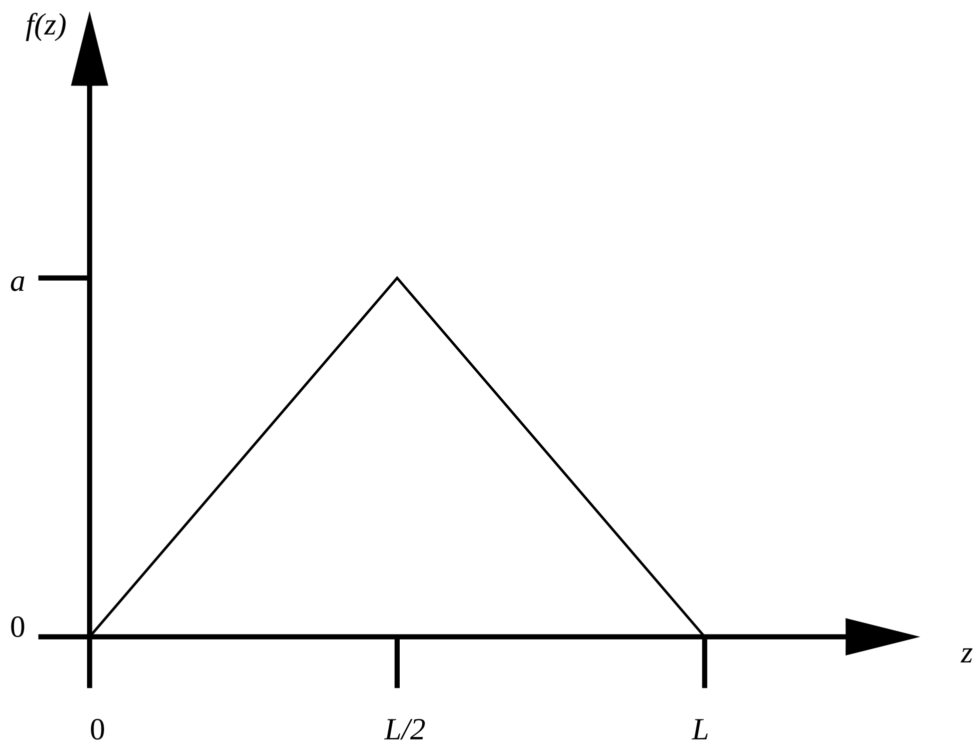
Some Sums

We can start with the result presented in B&B Page 175; we'll use $f(z) = s(z, 0) = s_0(z)$.



We are given that

$$f(z) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi z}{L}\right), \quad A_m = \frac{8a}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right).$$

With this form for A_m, it is easily seen that $A_m = 0$ for m even, and we can get a neat result in one fell swoop; note that

$$f(L/2) = a = \sum_{m=0}^{\infty} \frac{8a}{m^2\pi^2}\sin^2\left(\frac{m\pi}{2}\right) = \sum_{m \text{ odd}} \frac{8a}{m^2\pi^2},$$

so we have

Sum #1
$$\sum_{m \text{ odd}} \frac{1}{m^2} = \frac{\pi^2}{8}.$$

Now, note that

$$\frac{df}{dz} = \sum A_m \frac{m\pi}{L}\cos\left(\frac{m\pi z}{L}\right),$$

and that if you massage this properly (*i.e.*, shift the origin and turn Fig. 2.22 upside down), you get something like Equation 2.80. In any event,

$$\left.\frac{df}{dz}\right|_{z=0} = \frac{2a}{L} = \sum_{m=0}^{\infty} A_m \frac{m\pi}{L} \cos \left.\frac{m\pi z}{L}\right|_{z=0} = \sum_{m=0}^{\infty} \frac{8a}{m\pi L} \sin \frac{m\pi}{2},$$

so we have

Sum #2
$$\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right) = \frac{\pi}{4}.$$

Note that this is the Taylor Series for arctan(1) about $z = 0$. This is the age-old (at least as old as Newton) method of finding π numerically; it works, but it converges quite slowly. It also shows that π is irrational, if you had any doubts.

Next, we use a trick known to the mathematicians as the "energy theorem", a name which acknowledges their debt to physics. What we do is square $f(z)$, integrate from 0 to L, and compare the results. The function as graphed may be squared easily and integrated to give $a^2L/3$. In terms of the Fourier series expansion,

$$\begin{aligned}\int_0^L f^2\,dz &= \int_0^L \sum_{m,n=0}^{\infty} A_m\,A_n \sin\frac{m\pi z}{L} \sin\frac{n\pi z}{L}\,dz \\ &= \sum_{m,n=0}^{\infty} A_m\,A_n \int_0^L \sin\frac{m\pi z}{L} \sin\frac{n\pi z}{L}\,dz.\end{aligned}$$

As we have seen, the integral in the last expression vanishes if $n \neq m$, and is $L/2$ if $n = m$, so

$$\frac{a^2L}{3} = \frac{L}{2}\sum_{m=0}^{\infty} A_m^2 = \frac{64a^2}{2\pi^4} \sum_{m \text{ odd}} \frac{1}{m^4},$$

so we have

Sum #3
$$\sum_{m \text{ odd}} \frac{1}{m^4} = \frac{\pi^4}{96}.$$

We can still do more; we have $\sum_{\text{odds}} + \sum_{\text{evens}} = \sum_{\text{all}}$, and

$$\sum_{m \text{ even}} \frac{1}{m^4} = \sum_{\text{all } m} \frac{1}{(2m)^4} = \frac{1}{16} \sum_{\text{all } m} \frac{1}{m^4}, \quad \text{so}$$

$$\sum_{\text{odds}} \frac{1}{m^4} + \frac{1}{16}\sum_{\text{all } m} \frac{1}{m^4} = \sum_{\text{all } m} \frac{1}{m^4}, \quad \sum_{\text{all } m} \frac{1}{m^4} = \frac{16}{15} \sum_{m \text{ odd}} \frac{1}{m^4},$$

and we then have

Sum #4 $$\sum_{m=1}^{\infty} \frac{1}{m^4} = \frac{\pi^4}{90}.$$

This last sum will be useful later on when we do quantum mechanics; indeed, such sums, known as “Riemann-zeta functions” appear often in physics. You may note that from Sum #1 we can get

Sum #5 $$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}.$$

/mit/esg/pers/watko/8.03/notes/sums.tex