Problem 1. A microstrip line is made of a conducting strip placed on top of a grounded conducting plate, with an insulating sheet in between. The strip is usually gold or copper, plated onto the insulator, which is either alumina or fused quartz. Microstrip lines are used commonly in printed and integrated circuits that operate at high frequencies ($f \sim 1 - 10 \text{GHz}$). In practice the width $b$ of the strip is much larger than its distance $h$ to the grounded plate, so edge effects can be neglected.

Calculate the capacitance per unit length $C_0$, the inductance per unit length $L_0$, the speed of propagation $c$, and the characteristic impedance $Z_0$. Make sure you justify clearly (in words and using sketches) all the steps in your derivation. Hint: $Z_0 = (h/b) \sqrt{\mu/\epsilon}$.

Problem 2. Consider a realistic (lossy) transmission line in which the conductors have resistance per unit length $R_0$ [Ω m$^{-1}$] and the dielectric has conductance (inverse resistance) per unit length $G_0$ [S m$^{-1}$, where $S = $ siemens $= \Omega^{-1}$].

(a) Show that the equations for the voltage $\psi(x,t)$ and current $i(x,t)$ on the line are

$$\frac{\partial \psi}{\partial x} = -L_0 \frac{\partial i}{\partial t} - R_0 i, \quad \frac{\partial i}{\partial x} = -C_0 \frac{\partial \psi}{\partial t} - G_0 \psi.$$

(b) Now eliminate $i$ and show that the wave equation for $\psi$ becomes

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = (R_0 C_0 + G_0 L_0) \frac{\partial \psi}{\partial t} + R_0 G_0 \psi.$$

This is known as the telegraph equation.

Problem 3. A DC voltage generator is connected at one end ($x = 0$) of an ideal transmission line of length $L$ and characteristic impedance $Z_0$. At the other end ($x = L$) the line is terminated on a purely resistive load of impedance $Z_L = 3Z_0/2$. At $t = 0$ the voltage is switched on so that a step function of amplitude $V_0$ is applied at $x = 0$. The generator is matched to the line (i.e., its internal impedance is equal to $Z_0$), so that a wave incident on the generator will not be reflected.

(a) Write down the expressions for the voltage and current measured (i) at the end of the line ($x = L$), and (ii) at the midpoint ($x = L/2$), as a function of time for all times $t > 0$.

(b) What is the steady-state power dissipated in the load? In the entire system?

Problem 4. Consider the transverse vibrations of a string of length $L$, under tension $T$, and with mass per unit length $\mu$. The string is fixed at one end ($x = 0$) while at the other end ($x = L$) it is connected to a ring slipping on a rod, so that the ring is free to move in the transverse ($y$) direction.

At $t = 0$ the displacement $y(x, t)$ of the string is described by a stationary triangular pulse of width $h$ and height $y_0 \ll L$ centered on the midpoint of the string:

$$y(x, 0) \equiv y_0(x) = \begin{cases} 
0 & \text{for } 0 \leq x < (L - h)/2, \\
(y_0/h) [2x - L + h] & \text{for } (L - h)/2 \leq x < L/2, \\
(y_0/h) [L + h - 2x] & \text{for } L/2 \leq x < (L + h)/2, \\
0 & \text{for } (L + h)/2 \leq x \leq L,
\end{cases}$$

for $0 \leq x \leq L$. Due Friday, April 3
and \( \frac{\partial y}{\partial t}(x, 0) = 0 \).

(a) First assume an ideal boundary condition with an antinode at \( x = L \) (equivalent to assuming that the ring has negligible mass and that it is slipping without friction on the rod). Write down the solution \( y(x, t) \) as a superposition of two traveling waves. What is the period \( P \) [s] of the vibrations? Sketch the shape of the string at a few different times during the first cycle.

(b) Now, instead of assuming an ideal boundary condition with an antinode at \( x = L \), take into account the friction force on the ring. The boundary condition becomes

\[
-T \frac{\partial y}{\partial x} - b \frac{\partial y}{\partial t} = 0 \quad \text{at} \quad x = L.
\]

The first term is the vertical force exerted on the ring by the tension, while the second term is the friction force on the ring. In the limit where the mass of the ring goes to zero, the total force on the ring must be zero at all times. If \( b = 0.01 \sqrt{T/\mu} \), show that the energy \( E \) in the vibrating string is dissipated at an average rate described approximately by

\[
E(t) \simeq E_0 e^{-t/\tau},
\]

with \( \tau \simeq 50L/v \). Define and evaluate the \( Q \) factor of the string.