Lecture Notes for
Quantum Physics II & III
8.05 & 8.059
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4. Supplementary Notes on Time and Space Evolution of a Neutrino Beam

As an example of a two-state system in quantum mechanics, we discuss the oscillations of neutrinos produced in the decay of π^+; such oscillations would occur only if neutrinos had non-vanishing masses, which still has to be experimentally determined. The energy of the neutrinos in this case is mainly kinetic, since their velocity is close to c, the speed of light. The dynamics of relativistic particles is a subject for a Quantum Field Theory course, and Quantum Mechanics is not able to analyze it properly; nevertheless, in this particular case, we can deduce a sensible result by introducing some reasonable approximations.

We shall start from the time dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r};t) = H\,\psi(\mathbf{r};t) \tag{1}$$

and guess that the Hamiltonian for a relativistic free particle might be given by the following expression (resembling the classical relativistic energy)

$$H = \sqrt{\hat{\mathbf{p}}^2 c^2 + m^2 c^4} \tag{2}$$

so that in the Schrödinger picture we would have

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r};t) = \sqrt{-\hbar^2 c^2 \nabla^2 + m^2 c^4}\,\psi(\mathbf{r};t) \tag{3}$$

A square root of an operator gives us quite a lot of troubles; we shall instead iterate the above equation and exploit the fact that the LHS commutes with the RHS, obtaining the following

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi(\mathbf{r};t) = \left(-\hbar^2 c^2 \nabla^2 + m^2 c^4\right) \psi(\mathbf{r};t) \tag{4}$$

which is indeed one of the most famous (and early) equation of relativistic quantum mechanics: the Klein-Gordon equation. Moving the LHS to the RHS and dividing by $\hbar^2 c^2$, we get

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\psi\left(\mathbf{r};t\right) \;=\; 0 \tag{5}$$

after defining the d'Alambertian operator $\Box$

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \tag{6}$$

we are able to re-write the Klein-Gordon equation in a more compact and familiar way ($\hbar = c = 1$)

$$\left(\Box + m^2\right)\psi(x^\mu) = 0 \tag{7}$$

where $x^\mu = (ct, x, y, z)$.

We shall give here an argument in one-space and one-time dimension ($1+1$ dimension), but it is clear that the three-space and one-time dimension ($3+1$ dim) generalization is straightforward. In the $1+1$ dim counterpart of (4)

$$-\hbar^2\frac{\partial^2}{\partial t^2}\,\psi(x;t) = \left(-\hbar^2c^2\frac{\partial^2}{\partial x^2} + m^2c^4\right)\psi(x;t) \tag{8}$$

we separate the variables x and t by means of the usual *ansatz* (the RHS does not depend on t)

$$\psi(x;t) = \varphi(x)\phi(t) \tag{9}$$

After introducing a suitable constant E^2 (the square of the "eigenenergy"), we are led to two equations for the time and the spatial dependence; respectively

$$-\hbar^2\frac{\partial^2}{\partial t^2}\,\phi(t) = E^2\,\phi(t) \tag{10}$$

which is solved , for example, by[1]

$$\phi(t) \;=\; e^{-iEt/\hbar} \tag{11}$$

and

$$\left(-\hbar^2c^2\frac{\partial^2}{\partial x^2} + m^2c^4\right)\varphi(x) = E^2\,\varphi(x) \tag{12}$$

which we shall recast as follows

$$-\frac{\partial^2}{\partial x^2}\varphi(x) = \frac{1}{\hbar^2}\frac{\left(E^2 - m^2c^4\right)}{c^2}\,\varphi(x) \tag{13}$$

[1]Note that the other choice of sign, $\phi(t) = \exp(+iEt/\hbar)$ corresponds to *negative energy*, which is problematic. This peculiar feature of the Klein-Gordon equation is only resolved in quantum field theories that include both particles and antiparticles. For the present, we ignore the negative energy solutions.

Upon defining

$$p^2 = \frac{\left(E^2 - m^2c^4\right)}{c^2} \tag{14}$$

we are able to present it simply as

$$-\frac{\partial^2}{\partial x^2}\varphi(x) = \left(\frac{p}{\hbar}\right)^2 \varphi(x) \tag{15}$$

and solve it by means of well-known eigenfunctions

$$\varphi(x) \; = \; e^{ipx/\hbar} \tag{16}$$

Collecting the last expressions for ϕ and φ, we shall regard

$$\psi(x;t) \; = \; e^{ipx/\hbar}\, e^{-iEt/\hbar} \tag{17}$$

as the time evolution of the eigenfunction corresponding to the squared eigenenergy E^2.

Another way of discussing the time-evolution of the wavefunction consists in formally introducing a time-evolution operator

$$\hat{U}(t,t_o) = \exp\left[-i\sqrt{\hat{p}^2c^2 + m^2c^4}\,(t-t_o)/\hbar\right] \tag{18}$$

which in the Schrödinger representation reads

$$\hat{U}_S(t,t_o) = \exp\left[-i\sqrt{-\hbar^2c^2\partial^2/\partial x^2 + m^2c^4}\,(t-t_o)/\hbar\right] \tag{19}$$

so that the time-evolution of a generic wavefunction is given by

$$\langle x|\psi(t)\rangle \; = \; \langle x|\hat{U}(t)|\psi(0)\rangle \; = \; \hat{U}_S(t)\langle x|\psi(0)\rangle \; \equiv \; \hat{U}_S(t)\langle x|\psi\rangle_o \tag{20}$$

We shall now apply the previous arguments to the quantum-mechanical description of neutrino oscillations.

Muon (ν_μ) and electron neutrinos (ν_e) are related to the neutrino mass eigenstates (ν_1 and ν_2) by means of the following linear superpositions

$$\begin{aligned}\langle x|\nu_1\rangle_o &= \cos\vartheta\langle x|\nu_e\rangle_o - \sin\vartheta\langle x|\nu_\mu\rangle_o \\ \langle x|\nu_2\rangle_o &= \sin\vartheta\langle x|\nu_e\rangle_o + \cos\vartheta\langle x|\nu_\mu\rangle_o\end{aligned} \tag{21}$$

where the subscript denotes the initial time $t = 0$. Inverting the last equations and applying the time-evolution operator to both sides of the resulting equations leads us to express the time-dependent wavefunctions for the electron and muon neutrinos (eigenstates of the weak interaction) in terms of those ones for the free relativistic Hamiltonian eigenstates: ν_1 and ν_2

$$\langle x|\nu_e(t)\rangle = \hat{U}_S(t)\langle x|\nu_e\rangle_o = \cos\vartheta\,\langle x|\nu_1(t)\rangle + \sin\vartheta\,\langle x|\nu_2(t)\rangle$$

$$\langle x|\nu_\mu(t)\rangle = \hat{U}_S(t)\langle x|\nu_\mu\rangle_o = -\sin\vartheta\,\langle x|\nu_1(t)\rangle_o + \cos\vartheta\,\langle x|\nu_2(t)\rangle_o \tag{22}$$

We are discussing the case in which no electron neutrino is present at the initial time

$$\langle x|\nu_e\rangle_o = 0 \tag{23}$$

and the muon neutrino flux, originating from the decay of the π^+, is described by a wave packet such as

$$\langle x|\nu_\mu\rangle_o = \int \frac{dp}{\sqrt{2\pi\hbar}}\,\Phi(p)e^{ipx/\hbar} \tag{24}$$

where $\Phi(p)$ is a very well-localized function centered in p_o, the momentum transferred to ν_μ during the decay process. By replacing the last two conditions in (21), we get the initial wavefunctions for the eigenstates of the free Hamiltonian

$$\langle x|\nu_1\rangle_o = -\sin\vartheta \int \tfrac{dp}{\sqrt{2\pi\hbar}}\,\Phi(p)e^{ipx/\hbar}$$

$$\langle x|\nu_2\rangle_o = \cos\vartheta \int \tfrac{dp}{\sqrt{2\pi\hbar}}\,\Phi(p)e^{ipx/\hbar} \tag{25}$$

Let us discuss the propagation of ν_1 and omit the details for the case with ν_2 since they are very similar.

By applying here either the time-evolution operator or the argument for the decomposition in time-dependent eigenstates, we get

$$\langle x|\nu_1(t)\rangle = \hat{U}_S(t)\langle x|\nu_1\rangle_o$$

$$= -\sin\vartheta \int \tfrac{dp}{\sqrt{2\pi\hbar}}\,\Phi(p)e^{ipx/\hbar}e^{-ic\sqrt{p^2+m_1^2c^2}\,t/\hbar} \tag{26}$$

the expansion of the square root truncated to the first two terms reads

$$\sqrt{p^2+m^2c^2} \approx p + \frac{m^2c^2}{2p} \tag{27}$$

so that

$$\langle x|\nu_1(t)\rangle \approx -\sin\vartheta \int \frac{dp}{\sqrt{2\pi\hbar}}\;\; \Phi(p)e^{ipx/\hbar}e^{-ic(p+\frac{m_1^2c^2}{2p})t/\hbar} \tag{28}$$

Notice at this point that $\Phi(p)$ is peaked around p_o (as if it were a narrow Gaussian function) and causes the integrand to practically vanish outside a small neighbour of p_o. In this interval the oscillations of the phase $\exp\left(-i\frac{m_1^2c^3}{2\hbar p_o}t\right)$ are negligible with respect to $\exp\left[\frac{ip(x-ct)}{\hbar}\right]$; we shall therefore regard the former phase as a constant and take it out of the integration

$$\langle x|\nu_1(t)\rangle \approx\; -\sin\vartheta\, \exp\left(-i\frac{m_1^2c^3}{2\hbar p_o}t\right)\int \frac{dp}{\sqrt{2\pi\hbar}}\,\Phi(p)e^{ip(x-ct)/\hbar} \tag{29}$$

we may also say that, from a physical point of view, it has just been assumed that the neutrinos produced in the decay process are moving at the light speed.

Let us now introduce a more compact notation for the freely propagating wavepacket (with c velocity)

$$\psi_f(x-ct) \equiv \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}}\, \Phi(p) e^{ip(x-ct)/\hbar} \tag{30}$$

ψ_f satisfies the following normalization condition

$$\int_{-\infty}^{\infty} dx\, \left|\psi_f(x-ct)\right|^2 = 1 \tag{31}$$

and by means of it we can express the temporal evolution of the eigenstates of the free relativistic Hamiltonian as follows

$$\begin{aligned} \langle x|\nu_1(t)\rangle &\approx -\sin\vartheta\, \exp\left(-i\tfrac{m_1^2c^3}{2\hbar p_o}t\right)\psi_f(x-ct) \\ \langle x|\nu_2(t)\rangle &\approx \cos\vartheta\, \exp\left(-i\tfrac{m_2^2c^3}{2\hbar p_o}t\right)\psi_f(x-ct) \end{aligned} \tag{32}$$

If we wish to evaluate the probability of finding an electron neutrino (revealed for example through another weak interaction process) at a later time, we need the time-dependent wavefunction for ν_e as given in (22), that in the present case reads

$$\langle x|\nu_e(t)\rangle \approx \cos\vartheta\, \sin\vartheta\, \left[\exp\left(-i\frac{m_2^2c^3}{2\hbar p_o}t\right) - \exp\left(-i\frac{m_1^2c^3}{2\hbar p_o}t\right)\right]\, \psi_f(x-ct) \tag{33}$$

We shall now simply assume that the probability density of finding an electron neutrino at the position x at time t is given by

$$d\mathcal{P}_{\nu_e}(x;t) = \langle x|\nu_e(t)\rangle^* \langle x|\nu_e(t)\rangle = \left|\langle x|\nu_e(t)\rangle\right|^2 \tag{34}$$

If we sit at a distance x_o far from the neutrino source and wait for a ν_e to pass by in the time-interval dt about[2] $t_o = x_o/c$, then a length cdt of the wavefunction will cross the position x_o in the interval dt; hence the probability of observing an electron neutrino in the finite time-interval $\Delta\epsilon$ will be

$$\begin{aligned} \mathcal{P}_{\nu_e}\left(x_o; [t_o-\epsilon, t_o+\epsilon]\right) &= \textstyle\int_{t_o-\epsilon}^{t_o+\epsilon} cdt\, \left|\langle x_o|\nu_e(t)\rangle\right|^2 \\ &= (\cos\vartheta\, \sin\vartheta)^2 \textstyle\int_{t_o-\epsilon}^{t_o+\epsilon} cdt\, \left|\exp\left(-i\tfrac{m_2^2c^3}{2\hbar p_o}t\right) - \exp\left(-i\tfrac{m_1^2c^3}{2\hbar p_o}t\right)\right|^2 \left|\psi_f(x_o-ct)\right|^2 \\ &\approx (\cos\vartheta\, \sin\vartheta)^2 \left|\exp\left(-i\tfrac{m_2^2c^3}{2\hbar p_o c}x_o\right) - \exp\left(-i\tfrac{m_1^2c^3}{2\hbar p_o c}x_o\right)\right|^2 \\ &\qquad \textstyle\int_{t_o-\epsilon}^{t_o+\epsilon} cdt\, \left|\psi_f(x_o-ct)\right|^2 \end{aligned} \tag{35}$$

[2] t_o is approximately the moment we expect the maximum of the neutrino wavefunction to be at x_o.

Note that in the last approximation the slow-oscillating phase has been moved out of the integration as we did above. Neglecting the "tails" of the very well-localized wavepacket, which is essentially zero except when $t = t_o$ because of the form of ψ_f, we have

$$\int_{t_o-\epsilon}^{t_o+\epsilon} cdt \ |\psi_f(x_o - ct)|^2 \approx 1 \tag{36}$$

and the familiar result for the probability of observing an electron neutrino at the distance x_o is recovered by replacing $E \approx p_o c$

$$\begin{aligned} \mathcal{P}_{\nu_e}(x_o) &= (\cos\vartheta \ \sin\vartheta)^2 \left|\exp\left(-i\tfrac{m_2^2c^3}{2\hbar p_o c}x_o\right) - \exp\left(-i\tfrac{m_1^2c^3}{2\hbar p_o c}x_o\right)\right|^2 \\ &\approx (\cos\vartheta \ \sin\vartheta)^2 \left|\exp\left(-i\tfrac{m_2^2c^3}{2\hbar E}x_o\right) - \exp\left(-i\tfrac{m_1^2c^3}{2\hbar E}x_o\right)\right|^2 \end{aligned} \tag{37}$$