

SUPPLEMENTARY NOTES ON SOLVING THE RADIAL WAVE EQUATION USING OPERATOR METHODS

**Alternative title: An Introduction to Supersymmetric Quantum
Mechanics**

1 Introduction

In lecture this week we reduced the problem of finding the energy eigenstates and eigenvalues of hydrogenic atoms to finding the eigenstates and eigenvalues of the Coulomb Hamiltonians

$$H_\ell^{\text{Coul}} = -\frac{d^2}{dx^2} + V_\ell(x) \quad (1)$$

where

$$V_\ell(x) = \frac{\ell(\ell+1)}{x^2} - \frac{1}{x}. \quad (2)$$

(You will review this procedure on a Problem Set.) That is, we are looking for the eigenstates $u_{\nu\ell}(x)$ and eigenvalues $-\kappa_{\nu\ell}^2$ satisfying

$$H_\ell^{\text{Coul}} u_{\nu\ell}(x) = -\kappa_{\nu\ell}^2 u_{\nu\ell}(x). \quad (3)$$

Here the dimensionless variable x and dimensionless eigenvalue constants $\kappa_{\nu\ell}$ are related to r and the energy by

$$r = x\left(\frac{a}{2Z}\right), \quad E_{\nu\ell} = -\frac{2\mu e^4 Z^2}{\hbar^2} \kappa_{\nu\ell}^2, \quad (4)$$

where $a = \hbar^2/(me^2)$ is the Bohr radius, the nuclear charge is Ze , and $\mu = (1/m + 1/m_N)^{-1}$ is the reduced mass in terms of the electron mass m and nuclear mass m_N . Even though the Hamiltonians H_ℓ^{Coul} secretly all come from a single three-dimensional problem, for our present purposes it is best to think of them as an infinite number of different one-dimensional Hamiltonians, one for each value of ℓ , each with different potentials $V_\ell(x)$. The Hilbert space for these one-dimensional Hamiltonians consists of complex functions of $x \geq 0$ that vanish for $x \rightarrow 0$. The label ν distinguishes the different energy eigenstates and eigenvalues for a given H_ℓ^{Coul} .

Many texts, including Griffiths and Cohen-Tannoudji, teach you how to solve the Schrödinger equation (3) using standard, but somewhat tedious, power series methods. In this case the

solution to this second-order differential equation leads to a recursion relation that generates the Laguerre polynomials (see Griffiths, §4.2). In these notes, we shall investigate a different method, somewhat analogous to the technique we used to solve the harmonic oscillator earlier in 8.05. We shall employ operator methods that allow us to first find the ground state for each H_ℓ by solving a simple first order differential equation, and then construct all the excited states by acting with operators on the ground states. The key difference is that whereas when we analyzed the harmonic oscillator previously, the operator a^\dagger connected eigenstates of the *same* Hamiltonian with *different* energy eigenvalues, the operator \mathcal{A}^\dagger that we construct below will connect eigenstates of *different* Hamiltonians (namely H_ℓ and $H_{\ell+1}$) with the *same* energy eigenvalue. In this way, we shall derive a striking regularity of the hydrogen spectrum: the fact that the ν th energy eigenvalue of H_ℓ is the same as the $(\nu - 1)$ st energy eigenvalue of $H_{\ell+1}$.

In Section 2, I will present a very general method of relating the eigenstates and eigenvalues of pairs of one-dimensional Hamiltonians $H^{(1)}$ and $H^{(2)}$, where these Hamiltonians are not the same, but are both derived from the same function $\mathcal{W}(x)$. Then, in Section 3, we shall apply this method to solve the harmonic oscillator problem yet again. This is a bit like using a steam-roller when a fly-swatter would do, but it will serve as a good warm up for the procedure used for the three dimensional examples. We will discuss a free particle in 3-dimensions with spherical coordinates in Section 4, the Coulomb problem in Section 5, and finally the isotropic harmonic oscillator in Section 6. In Section 7 I outline the general procedure used in the operator method to solve for the bound state energies and wavefunctions. Finally, in Section 8 (which is optional extra reading) I will explain why the general method is called “supersymmetric quantum mechanics.”

2 The Factorization, or Supersymmetric, Method

2.1 Generalities

We are interested in studying Hamiltonians of the form,

$$H = -\frac{d^2}{dx^2} + V(x) . \quad (5)$$

(Just as we did for the hydrogen problem, it is always possible to rescale x and the energy such that the $\hbar^2/2m$ disappears and everything in the Schrödinger equation is dimensionless.)

Now, define a pair of operators,

$$\begin{aligned} \mathcal{A} &= \frac{d}{dx} + \mathcal{W}(x) \\ \mathcal{A}^\dagger &= -\frac{d}{dx} + \mathcal{W}(x) \end{aligned} \quad (6)$$

where \mathcal{W} is a smooth function of x known as the “superpotential”.¹ Recall that $i\frac{d}{dx}$ is Hermitian, meaning that $(\frac{d}{dx})^\dagger = -\frac{d}{dx}$. Thus, \mathcal{A}^\dagger is indeed the adjoint of \mathcal{A} .

We can construct two Hermitian Hamiltonians from \mathcal{A} and \mathcal{A}^\dagger as follows:

$$\begin{aligned} H^{(1)} &\equiv \mathcal{A}^\dagger \mathcal{A} = -\frac{d^2}{dx^2} + \mathcal{W}^2(x) - \mathcal{W}'(x) \\ H^{(2)} &\equiv \mathcal{A} \mathcal{A}^\dagger = -\frac{d^2}{dx^2} + \mathcal{W}^2(x) + \mathcal{W}'(x) . \end{aligned} \quad (7)$$

It is now easy to show that the energy spectra of these two Hamiltonians are the same, except for possible zero energy states. Here’s the proof:

Suppose that

$$H^{(1)} \phi_n = \mathcal{A}^\dagger \mathcal{A} \phi_n = E_n^{(1)} \phi_n \quad (8)$$

meaning that ϕ_n is the n th eigenstate of $H^{(1)}$ with eigenvalue $E_n^{(1)}$. Then, operating with \mathcal{A} and regrouping, we find

$$\begin{aligned} \mathcal{A} \mathcal{A}^\dagger \mathcal{A} \phi_n &= E_n^{(1)} \mathcal{A} \phi_n \\ [\mathcal{A} \mathcal{A}^\dagger] [\mathcal{A} \phi_n] &= E_n^{(1)} [\mathcal{A} \phi_n] . \end{aligned} \quad (9)$$

Since $\mathcal{A} \mathcal{A}^\dagger = H^{(2)}$, we conclude that the state $[\mathcal{A} \phi_n]$ is an eigenstate of $H^{(2)}$ with the *same* eigenvalue $E_n^{(1)}$. The proof works the other way as well: if

$$H^{(2)} \psi_n = \mathcal{A} \mathcal{A}^\dagger \psi_n = E_n^{(2)} \psi_n, \quad (10)$$

meaning that ψ_n is an eigenstate of $H^{(2)}$ with eigenvalue $E_n^{(2)}$, then acting on both sides with \mathcal{A}^\dagger we find that $\mathcal{A}^\dagger \psi_n$ is an eigenstate of $H^{(1)}$ with the same eigenenergy.

The only exception to this identification is the possibility that $\mathcal{A} \phi_0 = 0$ or $\mathcal{A}^\dagger \psi_0 = 0$. These correspond to zero energy eigenstates of $H^{(1)}$ and $H^{(2)}$ respectively. For this case it is obvious that the proof does not go through. If one of the Hamiltonians has an eigenstate with zero energy, the other need not. We shall now see that in fact the other *cannot* have a state with zero eigenvalue.

Let us try to construct the wave functions for a zero energy eigenstate explicitly. Consider $H^{(1)}$. A zero energy state obeys the simple, first order differential equation,

$$\mathcal{A} \phi_0(x) = \frac{d\phi_0}{dx} + \mathcal{W}(x) \phi_0(x) = 0, \quad (11)$$

which has the solution

$$\phi_0(x) = \exp \left[- \int_0^x dy \mathcal{W}(y) \right] , \quad (12)$$

¹The same setup applies for 1-dimensional problems on $-\infty < x < \infty$ and 3-dimensional problems on $0 < x < \infty$. For problems in three dimensions we would have subscripts ℓ everywhere (H_ℓ , \mathcal{A}_ℓ , \mathcal{A}_ℓ^\dagger , \mathcal{W}_ℓ) to remind us which effective 1-dimensional problem we were discussing. For convenience we use notation suited to a simple 1-dimensional problem in this section.

where we have set the lower limit of the y integral to 0 for the present. A shift in the lower limit changes $\phi_0(x)$ by a multiplicative constant. We are free to choose this multiplicative constant as we like because it is the integration constant. However, to be an acceptable bound state, ϕ_0 must be normalizable. That is, we must have

$$\int_{-\infty}^{\infty} dx |\phi_0(x)|^2 < \infty . \quad (13)$$

So long as this is satisfied, we can always choose the lower limit of integration — ie the multiplicative constant — to obtain a correctly normalized state. Looking back at eq. (12) we see that ϕ_0 vanishes as $x \rightarrow \pm\infty$ only if $\int_0^x dy \mathcal{W}(y) \rightarrow \infty$ as $x \rightarrow \pm\infty$. So, $H^{(1)}$ will have a zero energy state only if $\mathcal{W}(x)$ falls rather slowly (or even grows) as $x \rightarrow \pm\infty$.

Similarly, a zero energy eigenstate of $H^{(2)}$ obeys $\mathcal{A}^\dagger \psi_0 = 0$ and has the form

$$\psi_0(x) = \exp \left[\int_0^x dy \mathcal{W}(y) \right]. \quad (14)$$

Comparing Eqs. (12) and (14) we see that if ϕ_0 is normalizable, ψ_0 is not, and vice versa. So only one of the two Hamiltonians, $H^{(1)}$ or $H^{(2)}$ can have a zero energy eigenstate.

Finally, note that all the eigenvalues of both $H^{(1)}$ and $H^{(2)}$ are non-negative. This follows from an argument that should now be familiar, we have

$$\langle H^{(1)} \rangle = \langle \psi | \mathcal{A}^\dagger \mathcal{A} | \psi \rangle \geq 0, \quad \langle H^{(2)} \rangle = \langle \psi | \mathcal{A} \mathcal{A}^\dagger | \psi \rangle \geq 0, \quad (15)$$

for any $|\psi\rangle$ since the bra corresponding to $\mathcal{A}|\psi\rangle$ is $\langle\psi|\mathcal{A}^\dagger$ etc. For $|\psi\rangle$ an energy eigenstate this implies that the eigenvalues are non-zero. Also, it implies that a zero energy eigenstate is the ground state of the corresponding Hamiltonian (for cases where the zero energy eigenstate exists).

To summarize:

- The spectra of $H^{(1)}$ and $H^{(2)}$ are identical and the corresponding eigenstates ϕ_n and ψ_n are related in a simple way, $\psi_n = \mathcal{A}\phi_n$ and $\phi_n = \mathcal{A}^\dagger\psi_n$, except for possible zero energy states.
- The zero energy eigenstates are simple functions of the “superpotential” $\mathcal{W}(x)$. Only one of the two Hamiltonians can have a normalizable zero energy eigenstate. If a zero energy eigenstate exists, it is the ground state of the corresponding Hamiltonian.

We thus have a technique that allows us to relate eigenstates of two *different* Hamiltonians with the same energy eigenvalue.

2.2 How do we use $H^{(1)}$ and $H^{(2)}$ to solve problems?

If you are interested in some Hamiltonian, call it $H^{(1)}$, it is always possible to construct a superpotential \mathcal{W} from which $H^{(1)}$ can be generated as in Eq. (7). (We will not need this construction below, so I leave it for you to think about how to do this.) From \mathcal{W} , one can always construct $H^{(2)}$, whose eigenstates and eigenvalues are then related to those of $H^{(1)}$ in the sense that we have derived above. Thus, the existence of pairs of Hamiltonians related in this way is completely generic.

What is not generic is the possibility that both $H^{(1)}$ and $H^{(2)}$ may be related to the same physical problem. The facts that we have presented in the previous section will become very useful in cases when we have an additional relationship between $H^{(1)}$ and $H^{(2)}$, such as

$$H^{(2)} = H^{(1)} + \Delta \tag{16}$$

where Δ is a constant with no \hat{x} or \hat{p} dependence. A simple example which we discuss in the next section is the 1-dimensional harmonic oscillator, where the harmonic oscillator Hamiltonian $H^{\text{HO}} = H^{(1)} + 1 = H^{(2)} - 1$. This additional relationship together with the correspondence between the $H^{(1)}$ and $H^{(2)}$ spectra allow us to reproduce the energy levels of the harmonic oscillator. For three dimensions the analogous relationships are usually a bit more complicated. We will work through a few three dimensional examples and then extract the general features of this approach at the end.²

Our primary examples will be cases where an equation like (16) exists, however the relation between spectra for $H^{(1)}$ and $H^{(2)}$ can be useful even if this is not the case. You will see one example of this type in lecture, where the spectrum of $H^{(1)}$ will be much easier to solve directly, and then immediately gives us the spectrum for a more complicated Hamiltonian $H^{(2)}$. There is also no guarantee that faced with finding the solution to a radial problem that the supersymmetric method described in these notes will apply. However, it does work for a large number of cases.³

²Although I do know of several other universities where the factorization method is taught, most undergraduate curricula stick with solving the radial Schrödinger equation using the more traditional power series method. I first learned about the factorization method from Ohanian when I was an undergraduate studying quantum mechanics at the University of Manitoba in Canada. Schrödinger was the first person — in 1940 — to solve the hydrogen atom essentially by the method of these notes: he “factorized the Hamiltonian” into the product of an \mathcal{A} and an \mathcal{A}^\dagger and then used operator methods to get the eigenvalues and eigenstates. (Schrödinger did not call what he was doing supersymmetric quantum mechanics, because supersymmetry itself was still 30 years in the future at that point. See Section 8 of these notes for an introduction to supersymmetry, and an explanation of why these methods are now called supersymmetric quantum mechanics.) Given the elegance, simplicity, and long history of the method, it really surprises me that most texts present the much more painful solution to the Coulomb problem via Schrödinger’s second-order differential equation and the derivation of a recursion relation that generates the Laguerre polynomials. Go figure.

³L. Infeld and T.E. Hull have categorized the classes of potentials which have a supersymmetric solution

Reading on the factorization method can also be found in Chapters 6 and 8 of Ohanian. Note that Ohanian does not emphasize the paired nature of $H^{(1)}$ and $H^{(2)}$, and his notation is a bit different than the one used in lecture and used here, for example his η and η^\dagger are like our \mathcal{A} and \mathcal{A}^\dagger . In Chapter 8 Ohanian works through three important examples in more detail than you will find in these notes, namely the isotropic harmonic oscillator, the hydrogen atom, and a free particle in spherical coordinates.

3 The Harmonic Oscillator: $\mathcal{W}(x) = x$

The supersymmetric method discussed in the last section can be used to solve the harmonic oscillator. The exercise is worth investigating to make sure we understand the basic ideas. It is, however, a little too simple, in the sense that the two “different” Hamiltonians we study will not be very different.

The harmonic oscillator Schrödinger equation,

$$\left\{ -\frac{\hbar^2}{2m} \frac{d^2}{dy^2} + \frac{1}{2} m \omega^2 y^2 \right\} \Psi = E \Psi \quad (17)$$

can be scaled to dimensionless units

$$H^{\text{HO}} \Psi = \left(-\frac{d^2}{dx^2} + x^2 \right) \Psi = \beta \Psi \quad (18)$$

where $x = y \sqrt{m\omega/\hbar}$ and $\beta = 2E/\hbar\omega$. To produce a potential proportional to x^2 we need a superpotential linear in x . Consider

$$\mathcal{W}(x) = x, \quad (19)$$

from which we obtain

$$\begin{aligned} \mathcal{A} &= \frac{d}{dx} + x \quad \text{and} \quad \mathcal{A}^\dagger = -\frac{d}{dx} + x \\ H^{(1)} &= \mathcal{A}^\dagger \mathcal{A} = -\frac{d^2}{dx^2} + x^2 - 1 = H^{\text{HO}} - 1 \\ H^{(2)} &= \mathcal{A} \mathcal{A}^\dagger = -\frac{d^2}{dx^2} + x^2 + 1 = H^{\text{HO}} + 1 \end{aligned}$$

so

$$H^{(2)} = H^{(1)} + 2 \quad (20)$$

Step i). $H^{(1)}$ has a zero energy eigenstate, $H^{(1)}\phi_0 = 0$, because

$$\mathcal{A}\phi_0(x) = \left(\frac{d}{dx} + x \right) \phi_0(x) = 0$$

has a normalizable solution

in a mathematical paper which can be found in Reviews of Modern Physics, Vol.23, Number 1 (1951). The table at the end of their paper gives 31 different cases.

$$\phi_0(x) \propto \exp(-x^2/2). \quad (21)$$

Comparing $H^{(1)}$ and H^{HO} , we see that ϕ_0 is an eigenstate of H^{HO} with eigenvalue $\beta = 1$, *i.e.* $E = \hbar\omega/2$. $H^{(2)}$ does not have a normalizable zero energy state.

Step ii). We now use (20), namely $H^{(2)} = H^{(1)} + 2$:

$$H^{(2)}\phi_0 = H^{(1)}\phi_0 + 2\phi_0 = 2\phi_0. \quad (22)$$

So ϕ_0 ($\equiv \psi_1$) is an eigenstate of $H^{(2)}$ with eigenvalue 2. Then, according to the general properties we proved in Section 2, $\phi_1 \equiv \mathcal{A}^\dagger\psi_1$ must be an eigenstate of $H^{(1)}$ with the same eigenvalue. So $\mathcal{A}^\dagger\phi_0$ is an eigenstate of H^{HO} with eigenvalue $\beta = 3$, *i.e.* with $E = 3\hbar\omega/2$.

Step iii), repeat step ii) starting with ϕ_1 . All of the eigenenergies and eigenstates of the harmonic oscillator can be constructed by bouncing back and forth between $H^{(1)}$ and $H^{(2)}$ in this way: start with an eigenstate of $H^{(1)}$; use (20) to find an eigenstate of $H^{(2)}$ with eigenvalue β increased by 2; act with \mathcal{A}^\dagger to obtain a new eigenstate of $H^{(1)}$; repeat ad infinitum.

Despite the presentation, you should be able to determine that we have really not done anything new here. Choosing $\mathcal{W} = x$ yields $H^{(1)}$ and $H^{(2)}$ that differ only by a constant shift in the energy. In this overly simple example, therefore, finding an operator \mathcal{A}^\dagger which relates eigenstates of the different Hamiltonians $H^{(2)}$ and $H^{(1)}$ with the same eigenvalue is not a new story, as it is equivalent to finding an operator that relates eigenstates of the same Hamiltonian whose energy eigenvalues differ by that constant shift in the energy. You can confirm that \mathcal{A}^\dagger is just the familiar a^\dagger , multiplied by $\sqrt{2}$.

However, for other potentials this general method is very powerful. You can pick any arbitrarily complicated $\mathcal{W}(x)$, and from it find two Hamiltonians with *very* different potentials whose eigenstates and eigenvalues are related. You will work through one example on Problem Set 11. The most impressive example I have come across is illustrated in Figure 1.⁴ Here, $H^{(1)}$ describes a particle in the double-well potential shown, and $H^{(2)}$ describes a particle in the single-well potential. Two rather different looking potentials. And yet, because

⁴This beautiful example is described in the review paper by F. Cooper, A. Khare and U. Sukhatme, called ‘‘Supersymmetry and quantum mechanics,’’ to be found as Phys. Rept. **251**, 267 (1995) or on the web at [<http://arxiv.org/abs/hep-th/9405029>]. The figure is their Figure 4.3. The superpotential $\mathcal{W}(x)$ that generates these two potentials is known only as an infinite series, not in closed form. For this reason, we will do nothing with this example beyond admiring the figure. The review by Cooper, Khare and Sukhatme (whom I shall call CKS) is a very nice introduction to supersymmetric quantum mechanics, which is the method that we are learning to use in these notes. It is clearly written, and much of it should be accessible to you if you work your way through it carefully. Nevertheless, the methods, results, and connections to other areas of physics that CKS describe go well beyond what we will cover in 8.05. In 8.06, all of you will write term papers. I can easily imagine several different term paper topics emerging from careful study of CKS’ review.

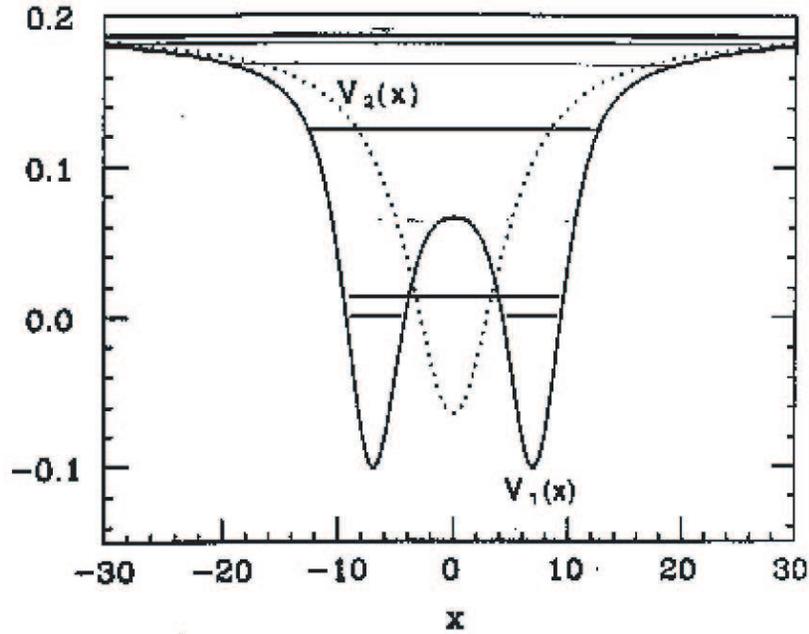


Figure 1: Two different potentials with the eigenvalues and eigenstates related as described in the text.

they are derived from the same superpotential $\mathcal{W}(x)$, all but one eigenvalue of $H^{(1)}$ is also an eigenvalue of $H^{(2)}$, the only exception being the ground state of the double well potential, which is a zero energy state. Even though their wave functions look rather different, each eigenstate of $H^{(2)}$ can be obtained from the corresponding eigenstate of $H^{(1)}$ by acting with \mathcal{A} .

4 Free Particle in Spherical Coordinates:

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The wave equation for a free particle of energy E in three dimensions can be written as $\psi_{E\ell m}(r, \theta, \phi) = \frac{1}{r}u_{E\ell}(r)Y_{\ell m}(\theta, \phi)$. Here $u_{E\ell}(r)$ is the radial wavefunction and satisfies the radial Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)\hbar^2}{2mr^2}\right)u_{E\ell}(r) = E u_{E\ell}(r). \quad (23)$$

Switching to dimensionless variables, we let $z = kr$ where $k^2 = 2mE/\hbar^2$. This gives

$$H_\ell^{\text{free}} u_\ell(z) = u_\ell(z), \quad H_\ell^{\text{free}} = -\frac{d^2}{dz^2} + \frac{\ell(\ell+1)}{z^2}, \quad (24)$$

where now the E dependence is all in the variable z . The solutions to this differential equation are given by spherical Bessel functions, $j_\ell(z)$ and spherical Neumann functions $\eta_\ell(z)$ and are discussed in most textbooks (including Griffiths in §4.1). It is the solution $u_\ell(z) = zj_\ell(z)$ that we want since it is finite at $r = 0$. Rather than studying the properties of these functions, we will instead solve (24) using the supersymmetric method.

The superpotential that does the job is $\mathcal{W}_\ell = -(\ell+1)/z$, and this gives⁶

$$\mathcal{A}_\ell = \frac{d}{dz} - \frac{(\ell+1)}{z}, \quad \mathcal{A}_\ell^\dagger = -\frac{d}{dz} - \frac{(\ell+1)}{z}. \quad (25)$$

From these we find with a bit of algebra that

$$\begin{aligned} H_\ell^{(1)} &= \mathcal{A}_\ell^\dagger \mathcal{A}_\ell = -\frac{d^2}{dz^2} + \frac{\ell(\ell+1)}{z^2} = H_\ell^{\text{free}}, \\ H_\ell^{(2)} &= \mathcal{A}_\ell \mathcal{A}_\ell^\dagger = -\frac{d^2}{dz^2} + \frac{(\ell+1)(\ell+2)}{z^2} = H_{\ell+1}^{\text{free}}, \end{aligned} \quad (26)$$

so the two Hamiltonians are related, $H_\ell^{(2)} = H_{\ell+1}^{(1)}$.

Let's pause for a minute and think about the equation we are trying to solve, (24). This is an eigenvalue equation where we already know the eigenvalue; it is one. This situation is special to the case of the free particle which has a continuum of possible energies E , and you should compare this example with the bound state examples treated in the next two sections. In the case of bound state problems where the energies are discrete we would need

⁵Even though this example is simpler than the two that follow in Sections 5 and 6, because this example concerns unbound states with unfamiliar wave functions (spherical Bessel functions) rather than bound states with familiar wave functions. You can skip this section now and read it only after Sections 5 and 6 if you like; understanding Sections 5 and 6 does not require understanding Section 4.

⁶At this point some of you will ask, given H how do I figure out what $\mathcal{W}(z)$ should be? Or equivalently, how do I determine what \mathcal{A} and \mathcal{A}^\dagger are? Ohanian discusses some strategies for doing this.

to find both the eigenvalues and eigenvectors of the radial equation (rather than just the eigenvectors). The free particle example is also special because it is a case where neither $H_\ell^{(1)}$ nor $H_\ell^{(2)}$ has a normalizable eigenstate with zero eigenvalue.⁷ The eigenstates of $H^{(1)}$ with zero eigenvalues are $z^{\ell+1}$ or $z^{-\ell}$, so they either blow up at $z = \infty$ or at $z = 0$. We want to determine the single eigenstate of $H_\ell^{(1)}$ with eigenvalue one for each ℓ . The supersymmetric method can be used to do this.

Let us first find $u_{\ell=0}(z)$. In this case (24) is just $-u_0'' = u_0$, and the solution that vanishes at $z = 0$ is

$$u_0(z) = \sin(z). \quad (27)$$

Now suppose that we have a solution for u_ℓ , then

$$u_{\ell+1}(z) = \mathcal{A}_\ell u_\ell(z) \quad (28)$$

is a solution of $H_{\ell+1}^{(2)} = H_{\ell+1}^{\text{free}}$. So given our solution in (27) we can generate all solutions for $\ell > 0$ using (28) and \mathcal{A}_ℓ . The algebra which shows that (28) is correct is the same as in (9) above. Start with $H_\ell^{\text{free}} u_\ell = u_\ell$ and act from the left with \mathcal{A}_ℓ to get

$$[\mathcal{A}_\ell \mathcal{A}_\ell^\dagger][\mathcal{A}_\ell u_\ell] = [\mathcal{A}_\ell u_\ell], \quad (29)$$

which shows that $H_{\ell+1}^{\text{free}}[\mathcal{A}_\ell u_\ell] = [\mathcal{A}_\ell u_\ell]$. In a similar fashion we can work out that $u_\ell(z) = \mathcal{A}_\ell^\dagger u_{\ell+1}(z)$. Thus \mathcal{A}_ℓ is acting like a raising operator for ℓ , and \mathcal{A}_ℓ^\dagger is acting like a lowering operator.⁸ Note however, that they raise and lower ℓ only in the space of radial wavefunctions and not for the angular $Y_{\ell m}$'s.

Let us finish this section by finding $u_1(z)$ and $u_2(z)$:

$$u_1(z) = \mathcal{A}_0 u_0(z) = \left(\frac{d}{dz} - \frac{1}{z} \right) \sin(z) = \cos(z) - \frac{\sin(z)}{z}, \quad (30)$$

$$u_2(z) = \mathcal{A}_1 u_1(z) = \left(\frac{d}{dz} - \frac{2}{z} \right) \left(\cos(z) - \frac{\sin(z)}{z} \right) = -\frac{3 \cos(z)}{z} + \left(\frac{3}{z^2} - 1 \right) \sin(z), \quad (31)$$

which you can confirm (for example by comparison with Griffiths) are indeed $z j_1(z)$ and $z j_2(z)$, with $j_\ell(z)$ the spherical Bessel functions. You can also confirm that u_0 , u_1 and u_2 all vanish at $z = 0$, ie at $r = 0$.

⁷Actually, the acceptable eigenstates will only have δ -function normalization.

⁸It is natural to ask, why in (28) is \mathcal{A}_ℓ acting like a raising operator for ℓ rather than \mathcal{A}_ℓ^\dagger ? Was this just a bad choice of notation? In fact it was deliberate, for bound state solutions we take \mathcal{A}_ℓ^\dagger to lower ℓ but to increase the number of radial nodes in the wavefunction. You can also ask how acting with an \mathcal{A}_ℓ^\dagger effects the energy of the state. This depends on the potential. We just saw that \mathcal{A}_ℓ^\dagger keeps the energy the same for a free particle, and we will see that it keeps the energy the same for Hydrogen but that it raises the energy for the isotropic Harmonic oscillator.

5 The Coulomb Potential: $\mathcal{W}_\ell = -\frac{(\ell+1)}{x} + \frac{1}{2(\ell+1)}$

To study the Coulomb problem, the superpotential has to be chosen to reproduce a $1/r$ potential as well as the angular momentum barrier. Consider

$$\mathcal{W}_\ell = -\frac{(\ell+1)}{x} + \frac{1}{2(\ell+1)}, \quad (32)$$

which is a different \mathcal{W} for every value of ℓ .⁹ From these \mathcal{W}_ℓ 's, we construct the operators

$$\mathcal{A}_\ell = \frac{d}{dx} - \frac{\ell+1}{x} + \frac{1}{2(\ell+1)}, \quad \mathcal{A}_\ell^\dagger = -\frac{d}{dx} - \frac{\ell+1}{x} + \frac{1}{2(\ell+1)}, \quad (33)$$

and from them we construct a pair of Hamiltonians for each value of ℓ :

$$\begin{aligned} H_\ell^{(1)} &= \mathcal{A}_\ell^\dagger \mathcal{A}_\ell = -\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} - \frac{1}{x} + \frac{1}{4(\ell+1)^2} = H_\ell^{\text{Coul}} + \frac{1}{4(\ell+1)^2} \\ H_\ell^{(2)} &= \mathcal{A}_\ell \mathcal{A}_\ell^\dagger = -\frac{d^2}{dx^2} + \frac{(\ell+1)(\ell+2)}{x^2} - \frac{1}{x} + \frac{1}{4(\ell+1)^2} = H_{\ell+1}^{\text{Coul}} + \frac{1}{4(\ell+1)^2}. \end{aligned} \quad (34)$$

The Hamiltonian for the Coulomb problem, H_ℓ^{Coul} , was given in Eq. (1), and we have found that both $H^{(1)}$ and $H^{(2)}$ are related to it. Furthermore we find that

$$H_\ell^{(2)} = H_{\ell+1}^{(1)} + \frac{1}{4(\ell+1)^2} - \frac{1}{4(\ell+2)^2}. \quad (35)$$

This looks very promising. By applying the results we proved in general in Section 2, we are going to obtain relations between eigenstates and eigenvalues of $H_\ell^{(1)}$ and $H_\ell^{(2)}$, and thus between eigenstates and eigenvalues of H_ℓ^{Coul} and $H_{\ell+1}^{\text{Coul}}$, for any value of ℓ . We'll then "stitch all these relations together" and obtain all the energy eigenstates and eigenvalues for each value of ℓ . Let's now do this, slowly and explicitly.

Step i). As before, let's begin by looking for a state annihilated by \mathcal{A}_ℓ :

$$\mathcal{A}_\ell \phi_{0\ell}(x) = \left[\frac{d}{dx} - \frac{\ell+1}{x} + \frac{1}{2(\ell+1)} \right] \phi_{0\ell}(x) = 0. \quad (36)$$

First order ordinary differential equations are easy to solve, and this one is no exception. The solution is (up to a normalization constant),

$$\begin{aligned} \phi_{0\ell}(x) &= e^{-\int_0^x \mathcal{W}(y) dy} \\ &= e^{(\ell+1) \ln x - \frac{x}{2(\ell+1)}} \\ &= x^{\ell+1} e^{-\frac{x}{2(\ell+1)}}, \end{aligned} \quad (37)$$

⁹If you have read Section 4, you recognize the first term in \mathcal{W}_ℓ from the analysis in that section of the angular momentum barrier without any $1/r$ potential. This section is however self-contained, because the $1/r$ -potential changes the physics qualitatively.

and this is a zero energy eigenstate of $H_\ell^{(1)}$,

$$H_\ell^{(1)} \phi_{0\ell}(x) = \mathcal{A}_\ell^\dagger \mathcal{A}_\ell \phi_{0\ell}(x) = 0. \quad (38)$$

You can verify that if you construct a $\psi(x)$ satisfying $\mathcal{A}_\ell^\dagger \psi(x) = 0$, this $\psi(x)$ grows exponentially at large x and so is not normalizable. Therefore, $H_\ell^{(2)}$ has no zero energy eigenstate.

Let's return to $\phi_{0\ell}(x)$, defined by Eq. (37). This is the wave function for the ground state of $H_\ell^{(1)}$ and has zero energy. Because $H_\ell^{(1)} = H_\ell^{\text{Coul}} + \frac{1}{4(\ell+1)^2}$, this means that $\phi_{0\ell}(x)$ is an eigenstate of H_ℓ^{Coul} with eigenvalue

$$-\kappa_{0\ell}^2 = -\frac{1}{4(\ell+1)^2}, \quad (39)$$

meaning that

$$E_{0\ell} = -\frac{2me^4 Z^2}{\hbar^2} \kappa_{0\ell}^2 = -\frac{me^4 Z^2}{2\hbar^2 (\ell+1)^2}. \quad (40)$$

You should recognize these as exactly the results we obtained in lecture when reviewing Hydrogen. $\phi_{0\ell}(x)$ is just $u_{0\ell}(x)$, the ground state for the ℓ th Coulomb Hamiltonian. Referring to Figure 2, so far we have found the lowest energy level for each ℓ .

Step ii). This time we are set up to find all the excited states. We have a normalizable state $\phi_{0,\ell+1}(x)$ that is a zero energy eigenstate of $H_{\ell+1}^{(1)}$, which in turn from equation (35) means that

$$H_\ell^{(2)} \phi_{0,\ell+1}(x) = \left\{ \frac{1}{4(\ell+1)^2} - \frac{1}{4(\ell+2)^2} \right\} \phi_{0,\ell+1}(x), \quad (41)$$

so that $\phi_{0,\ell+1}(x)$ is an eigenstate of $H_\ell^{(2)}$ with eigenvalue $\frac{1}{4} \left\{ \frac{1}{(\ell+1)^2} - \frac{1}{(\ell+2)^2} \right\}$. We now use the first result that we proved in general in Section 2 to conclude that $H_\ell^{(1)}$ also has an eigenvalue $\frac{1}{4} \left\{ \frac{1}{(\ell+1)^2} - \frac{1}{(\ell+2)^2} \right\}$ and a corresponding eigenstate given by acting with \mathcal{A}_ℓ^\dagger on $\phi_{0,\ell+1}(x)$. That is, if we define

$$\phi_{1\ell} \equiv \mathcal{A}_\ell^\dagger \phi_{0,\ell+1} \quad (42)$$

then we have shown that

$$H_\ell^{(1)} \phi_{1\ell} = \frac{1}{4} \left\{ \frac{1}{(\ell+1)^2} - \frac{1}{(\ell+2)^2} \right\} \phi_{1\ell}. \quad (43)$$

Recalling that $H_\ell^{(1)} = H_\ell^{\text{Coul}} + \frac{1}{4(\ell+1)^2}$, we conclude that

$$H_\ell^{\text{Coul}} \phi_{1\ell} = -\frac{1}{4(\ell+2)^2} \phi_{1\ell}, \quad (44)$$

so the states $\phi_{1\ell}$ have eigenvalues $-\kappa_{1\ell}^2 = -\frac{1}{4(\ell+2)^2}$.

Step iii), repeat. Applying the same argument over again, “doing to $\phi_{1\ell}$ what we did to $\phi_{0\ell}$ ”, we conclude that

$$\phi_{2\ell} \equiv \mathcal{A}_\ell^\dagger \phi_{1,\ell+1} = \mathcal{A}_\ell^\dagger \mathcal{A}_{\ell+1}^\dagger \phi_{0,\ell+2} \quad (45)$$

is an eigenstate of H_ℓ^{Coul} with eigenvalue

$$-\kappa_{2\ell}^2 = -\frac{1}{4(\ell+3)^2} . \quad (46)$$

And, in general,

$$\phi_{\nu\ell} = \mathcal{A}_\ell^\dagger \mathcal{A}_{\ell+1}^\dagger \cdots \mathcal{A}_{\ell+\nu-1}^\dagger \phi_{0,\ell+\nu} \quad (47)$$

is an eigenstate of H_ℓ^{Coul} with eigenvalue

$$-\kappa_{\nu\ell}^2 = -\frac{1}{4(\ell+\nu+1)^2} . \quad (48)$$

Switching back to the original variables, this means that the energy levels for Hydrogen are

$$E_{\nu\ell} = -\frac{me^4 Z^2}{2\hbar^2} \frac{1}{(\ell+\nu+1)^2} . \quad (49)$$

We now define Bohr's principal quantum number $n \equiv \ell + \nu + 1$ and obtain

$$E_n = -\frac{me^4 Z^2}{2\hbar^2} \frac{1}{n^2} = -(13.6 \text{ eV}) \frac{Z^2}{n^2} . \quad (50)$$

Since the integer $\nu \geq 0$ we have $\ell \leq n - 1$.

To conclude, we have managed to find all the eigenvalues and eigenstates of each of the H_ℓ^{Coul} Hamiltonian. We have learned that the ν th state with angular momentum ℓ has the same energy as the $(\nu-1)$ st state with angular momentum $(\ell+1)$. The operators \mathcal{A}_ℓ and \mathcal{A}_ℓ^\dagger turned these states into one another, but did not change the energy. Thus we have learned that the energy depends on ℓ and ν only through the combination $n \equiv \ell + \nu + 1$. In Figure 2 I show the energy levels for Hydrogen, along with the action of the operator \mathcal{A}_ℓ and \mathcal{A}_ℓ^\dagger on the states. The states in each column of this figure are the solutions to the eigenvalue problem for H_ℓ^{Coul} for a fixed ℓ . In thinking back about the procedure we used to construct these states, it is instructive to compare Figure 2 to a figure showing the energy levels for $H_\ell^{(1)}$ which I encourage you to draw. Recall that $H_\ell^{(1)}$ has the same eigenstates $\phi_{\nu\ell}$ as H_ℓ^{Coul} , but they are shifted by the ℓ dependent constant $\frac{1}{4(\ell+1)^2}$. Furthermore, each state $\phi_{0\ell}$ was an eigenstate of $H_\ell^{(1)}$ with eigenvalue zero.

Finally, it may be helpful for you if I explicitly construct one of the excited state wave functions. Let's do $u_{20}(x)$, namely the second excited state ($\nu = 2$) of the $\ell = 0$ Hamiltonian, which is one of the states with $n = 3$. We construct this beginning from the ground state with $\ell = 2$ as follows:

$$u_{20}(x) = \phi_{20}(x) = \mathcal{A}_0^\dagger \mathcal{A}_1^\dagger \phi_{02}(x) = \mathcal{A}_0^\dagger \mathcal{A}_1^\dagger \left(x^3 e^{-\frac{x}{6}} \right) \quad (51)$$

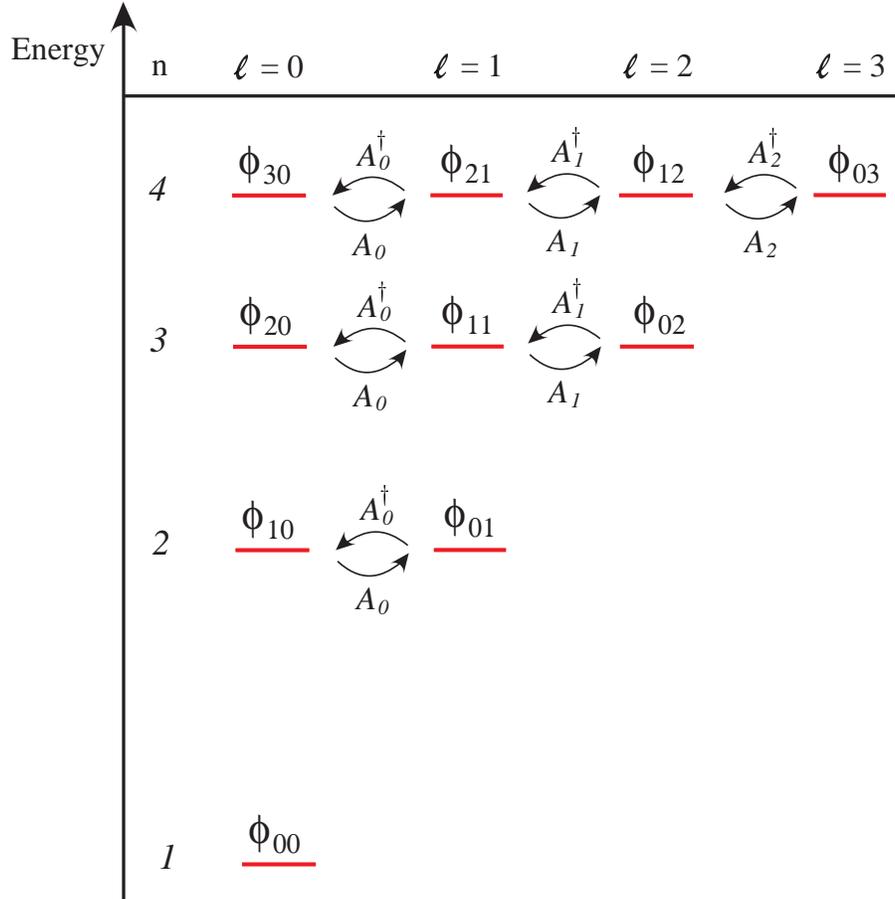


Figure 2: The energy levels of the hydrogen atom up to $n = 4$. Each level is labeled with the name of the corresponding eigenstate, $\phi_{\nu\ell}$. The states are organized in columns by their value of ℓ . The ν th state with angular momentum ℓ has the same energy as the $(\nu - 1)$ st state with angular momentum $\ell + 1$. That is, the energy depends only on the Bohr quantum number $n = \ell + \nu + 1$. The operators \mathcal{A}_ℓ^\dagger act on states in the $(\ell + 1)$ st column and turn them into the states in the ℓ th column with the same energy. The operators \mathcal{A}_ℓ act on states in the ℓ th column and, with one exception, turn them into the states in the $(\ell + 1)$ st column with the same energy. The one exception is that when \mathcal{A}_ℓ acts on the lowest state in the ℓ th column — the ground state of H_ℓ^{Coul} — it annihilates it.

where we have used the solution Eq. (37) for the ground state of the $\ell = 2$ Hamiltonian. The explicit expressions for the \mathcal{A}_ℓ^\dagger 's are:

$$\begin{aligned}\mathcal{A}_0^\dagger &= \frac{d}{dx} - \frac{1}{x} + \frac{1}{2} \\ \mathcal{A}_1^\dagger &= \frac{d}{dx} - \frac{2}{x} + \frac{1}{4}.\end{aligned}\tag{52}$$

After using $r = x(a/2Z)$ where $a \equiv \hbar^2/me^2$ is the Bohr radius, using $f(r) = (1/r)u(r)$, setting $Z = 1$, and upon ensuring that the state is correctly normalized, we find

$$f_{20}(r) = \frac{2}{\sqrt{27}}a^{-3/2} \left\{ 1 - \frac{2r}{3a} + \frac{2}{27} \left(\frac{r}{a} \right)^2 \right\} e^{-r/3a}.\tag{53}$$

(To restore the Z in (53), replace a by a/Z .) Since (53) is an $\ell = 0$ state, we complete the specification of the wave function for this particular excited state of hydrogen by multiplying by the angular wave function $Y_{00}(\theta, \phi) = 1/\sqrt{4\pi}$. Last of all, to avoid confusion when you are reading textbooks, please note that I have been labeling the radial wave functions by ν and ℓ , whereas it is conventional in most books to use n and ℓ .

6 The Isotropic Harmonic Oscillator: $\mathcal{W}_\ell = -\frac{(\ell+1)}{x} + x$

An isotropic Harmonic oscillator has the potential $V(r) = \frac{1}{2}m\omega^2r^2$ so this time the radial equation is

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} + \frac{1}{2}m\omega^2r^2 \right) u_{\nu\ell}(r) = E u_{\nu\ell}(r).\tag{54}$$

Making a change of variable to $x = r/b$ where $b^2 = \hbar/(m\omega)$, and to $\varepsilon_{\nu\ell} = 2E/(\hbar\omega)$ gives

$$H_\ell^{\text{ISO}} u_{\nu\ell}(x) = \varepsilon_{\nu\ell} u_{\nu\ell}(x), \quad H_\ell^{\text{ISO}} = -\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + x^2.\tag{55}$$

This time we take $\mathcal{W}_\ell = -(\ell+1)/x + x$ and therefore have

$$\mathcal{A}_\ell = \frac{d}{dx} - \frac{\ell+1}{x} + x, \quad \mathcal{A}_\ell^\dagger = -\frac{d}{dx} - \frac{\ell+1}{x} + x,\tag{56}$$

and

$$\begin{aligned}H_\ell^{(1)} &= \mathcal{A}_\ell^\dagger \mathcal{A}_\ell = -\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + x^2 - 2\ell - 3 = H_\ell^{\text{ISO}} - 2\ell - 3 \\ H_\ell^{(2)} &= \mathcal{A}_\ell \mathcal{A}_\ell^\dagger = -\frac{d^2}{dx^2} + \frac{(\ell+1)(\ell+2)}{x^2} + x^2 - 2\ell - 1 = H_{\ell+1}^{\text{ISO}} - 2\ell - 1,\end{aligned}\tag{57}$$

so that in this case we find

$$H_\ell^{(2)} = H_{\ell+1}^{(1)} + 4.\tag{58}$$

The steps for solving (55) are the same as for hydrogen.

Step i). Solve $H_\ell^{(1)}\phi_{0\ell}(x) = 0$ to find the ground states for each ℓ . The solution is

$$\phi_{0\ell}(x) = x^{\ell+1}e^{-x^2/2}, \quad (59)$$

where I left out the normalization constant.

Step ii). Find the first excited states for each ℓ . Using the relation (58) the state $\phi_{0\ell+1}$ is an eigenstate of $H_\ell^{(2)}$ with eigenvalue 4, and so $\phi_{1\ell} = \mathcal{A}_\ell^\dagger\phi_{0\ell+1}$ is an eigenstate of $H_\ell^{(1)}$ with eigenvalue 4.

Step iii), repeat. We find that $\phi_{\nu,\ell} = \mathcal{A}_\ell^\dagger \cdots \mathcal{A}_{\ell+\nu-1}^\dagger\phi_{0,\ell+\nu}$ is an eigenstate of $H_\ell^{(1)}$ with eigenvalue 4ν .

Translating back to the Hamiltonian H_ℓ^{ISO} using (57) we find $\varepsilon_{\nu\ell} = 2\ell + 4\nu + 3$ which gives the energy levels for the isotropic harmonic oscillator

$$E_{\nu\ell} = \hbar\omega\left(\ell + 2\nu + \frac{3}{2}\right). \quad (60)$$

In this case it seems most natural to define the principal quantum number $n = \ell + 2\nu$ in which case $E_n = \hbar\omega(n + 3/2)$ and since $\nu \geq 0$ we have $\ell \leq n$. The isotropic harmonic oscillator's energy spectrum is shown in Figure 3. Note how quickly we were able to obtain the entire spectrum of energy eigenvalues. Using the expressions for the operators \mathcal{A}_ℓ^\dagger we can also work out the corresponding eigenstates $\phi_{\nu,\ell}$.

7 General Approach for Three Dimensions

Now that we have gone through a few examples we can extract a general setup and procedure for solving for the spectrum and eigenstates of three dimensional radial problems which applies to a variety of examples. We started with a Hamiltonian H_ℓ whose bound state eigenvalues and eigenfunctions we wish to know, $H_\ell\phi_{\nu,\ell} = \varepsilon_{\nu,\ell}\phi_{\nu,\ell}$. The key was showing that it could be related to a Hamiltonian $H_\ell^{(1)}$ by

$$H_\ell = H_\ell^{(1)} + s_\ell, \quad (61)$$

where s_ℓ are constants chosen so $H_\ell^{(1)}$ has an eigenstate with zero eigenvalue, and that we also have

$$H_\ell^{(2)} = H_{\ell+1}^{(1)} + \Delta_\ell, \quad (62)$$

for some numbers $\Delta_\ell > 0$. Given equations like these, the steps for solving for the $H_\ell^{(1)}$ energies and eigenstates are

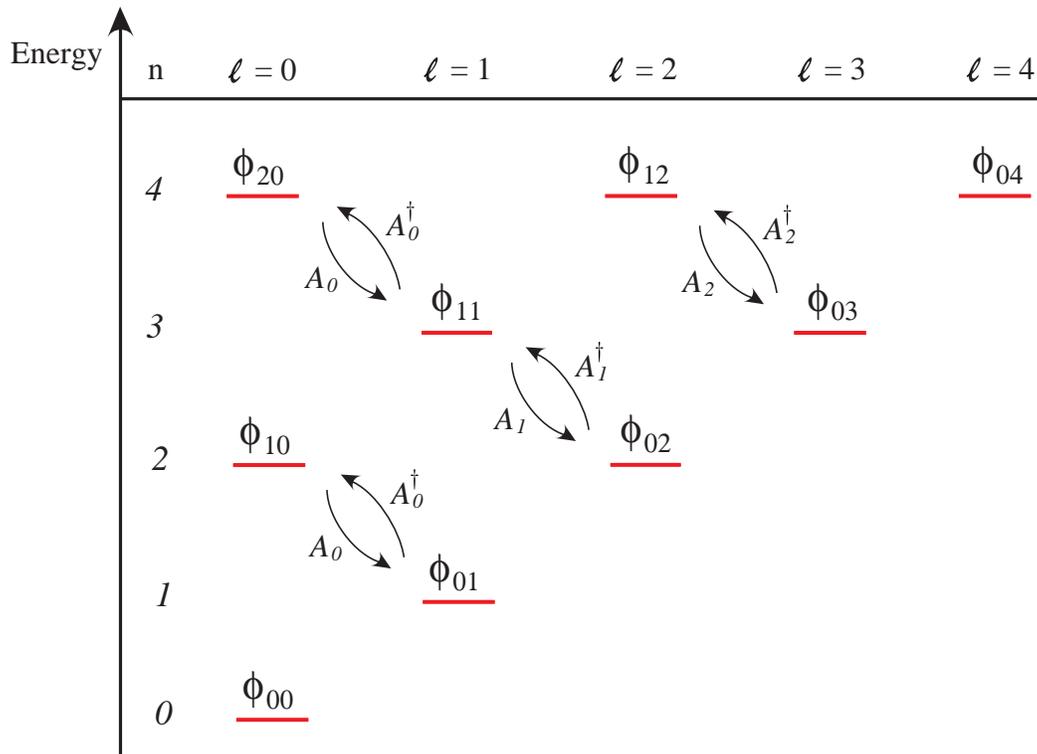


Figure 3: The energy levels of the isotropic harmonic oscillator up to $n = 4$, labeled by the eigenstates $\phi_{\nu\ell}$ where $n = \ell + 2\nu$. In this case the energy levels are equally spaced and the operators \mathcal{A}_ℓ^\dagger take us to a higher energy state just like the operator a^\dagger did for the simple 1-dimensional harmonic oscillator.

- i) Solve for the zero-energy ground state for each ℓ by solving the differential equation $\mathcal{A}_\ell \phi_{0,\ell} = 0$.
- ii) Given $\phi_{0,\ell}$ we now use the relation (62) to find an eigenstate of $H_\ell^{(2)}$ (namely $\phi_{0,\ell+1}$) with a larger energy, Δ_ℓ . Then we use the degeneracy of the $H_\ell^{(2)}$ and $H_\ell^{(1)}$ spectra to determine that $\phi_{1,\ell} = \mathcal{A}_\ell^\dagger \phi_{0,\ell+1}$ is an eigenstate of $H_\ell^{(1)}$ with this same energy.
- iii) repeat step ii) but start with $\phi_{1,\ell}$. Doing so over and over again we will generate the whole spectrum of eigenstates $\phi_{\nu,\ell} = \mathcal{A}_\ell^\dagger \cdots \mathcal{A}_{\ell+\nu-1}^\dagger \phi_{0,\ell+\nu}$ with larger and larger energies, $\Delta_\ell + \Delta_{\ell+1} + \dots + \Delta_{\ell+\nu-1}$.

Since the eigenvalues of $H_\ell^{(1)}$ and H_ℓ are related by the constant s_ℓ we can translate to the spectrum for H_ℓ (in fact we can do this at any point in the steps above). For H_ℓ we then find that the energy eigenstates are also $\phi_{\nu,\ell}$ with energy eigenvalues

$$\varepsilon_{\nu,\ell} = s_\ell + \sum_{j=0}^{\nu-1} \Delta_{\ell+j}, \quad (63)$$

and the sum only contributes for $\nu \geq 1$. These are the energy eigenvalues for the dimensionless Hamiltonian H_ℓ . As a final step we can undo any changes of variables to get the final dimensionful energies of interest, $E_{\nu,\ell}$.

8 Supersymmetric Quantum Mechanics

By now, you must be wondering why the general method we have used to find the hydrogen radial wave functions is called supersymmetric quantum mechanics. Some of you have probably heard a little bit about supersymmetry in popular accounts of modern theories of elementary particle physics, and if you have you likely do not see any connection between what you have heard and what we have done. It is not possible for me to make this connection completely rigorously, given that you have not yet even been introduced to bosons and fermions, leave apart to quantum field theory. I've decided to *try* to make the connection, though, in this last section of these notes. Reading this section is optional.

We know of many elementary particles with half-integer spin (electrons, muons, taus, quarks) and many with integer spin (photons, gluons, W's, Z's, gravitons). The half-integer spin particles are collectively known as "fermions" and the integer-spin particles are collectively known as "bosons". We shall learn more about what is significant about being a fermion or being a boson in the last section of 8.05. A "supersymmetric quantum field theory" is one in which for every fermion of a given mass, there is also a boson with the same mass.

It is easy to see that Nature is *not* supersymmetric. For example, photons are massless and we know of no corresponding massless fermions. We know of no boson with the same mass as the electron. In fact, we know of *no examples* of two elementary particles, one a fermion and the other a boson, with the same mass. One of the overarching themes of modern theoretical physics, however, is that the *laws* of nature are much more symmetric than the *state* of nature. Examples are easy to come by. All known laws of physics are rotationally invariant meaning that the laws of physics do not single out any particular direction. But, if you live immersed within a gravitational field, the up and down directions are most definitely singled out. There are many more examples of systems which choose states that “break” symmetries that are respected by the laws of nature. Theoretical physicists have therefore long wondered whether the laws of nature are supersymmetric, even though the state that we experience is not. The standard model of particle physics is not supersymmetric, but perhaps a more complete theory of nature will be. Supersymmetric theories resolve a number of outstanding puzzles that the standard model leaves unanswered (sorry that I have not backed up this statement for you.) Many theoretical physicists have therefore worked hard to understand how the laws of nature could be supersymmetric, while the state that we experience could “break supersymmetry”, meaning that all the unfamiliar particles, like the bosonic partner of the electron and the fermionic partner of the photon, are much heavier than the familiar particles. These ideas will be tested stringently by the Large Hadron Collider, a particle accelerator currently under construction in Geneva, Switzerland that, when it begins operations in 2007, will be able to search for new elementary particles up to about 10 times heavier than the top quark, which covers the mass regime where supersymmetry theorists predict that “superpartners” of many of our familiar particles are waiting to be discovered.

Let us now consider a supersymmetric theory, ignoring the possibility of supersymmetry breaking. That is, let’s think about a quantum field theory in which for every species of fermion with mass m , there is a boson with the same mass m . I will have to oversimplify this description; please forgive me. It is easiest to think about quantum field theory in momentum space. A state in the Hilbert space is specified by stating how many particles of a given species there are for each value of momentum (p_x, p_y, p_z) . What I want to show you is that if you restrict your attention only to particles with zero momentum, ie those at the point $(0, 0, 0)$ in momentum space, then supersymmetric quantum field theory becomes analogous to the supersymmetric quantum mechanics of these notes. Let’s consider a theory with one type of boson, and one type of fermion. And, let’s only consider particles with zero momentum. Then, the states in this zero momentum sector of the theory can be labeled $|n_b, n_f\rangle$, using the number of zero momentum bosons and fermions present in the state as labels. Since all these particles have zero momentum, if we ignore the possibility of

interactions between particles the energy of the state $|n_b, n_f\rangle$ is just $(n_b + n_f)mc^2$. From now on I will drop the factor of c^2 .

I now have to tell you some facts about fermions and bosons, which we will derive later in 8.05, but which should not sound surprising.

- When you add up all the spins (we'll see what this means precisely later) a state with any number of bosons and an even number of fermions has a total spin that is an integer, and thus is “bosonic”.
- When you add up all the spins, a state with any number of bosons and an odd number of fermions has a total spin that is a half-integer, and thus is “fermionic”.

Think about all possible bosonic states in the zero-momentum sector of the theory which we are describing. The bosonic state $|0, 0\rangle$ has energy 0. $|1, 0\rangle$ is also a bosonic state, and has energy m . $|2, 0\rangle$ and $|0, 2\rangle$ are bosonic states that both have energy $2m$. $|3, 0\rangle$ and $|1, 2\rangle$ are bosonic states with energy $3m$. Clearly, you can find bosonic states with energies $0, m, 2m, 3m, 4m, \dots$. Now, think of all possible fermionic states in the zero-momentum sector of the theory. The state $|0, 1\rangle$ has energy m . The state $|1, 1\rangle$ has energy $2m$. The fermionic states $|2, 1\rangle$ and $|0, 3\rangle$ both have energy $3m$. Clearly, you can find fermionic states with energies $m, 2m, 3m, 4m, \dots$ ¹⁰

After generalization to a theory with more than one species of boson and fermion and after the possibility of interactions between particles is taken into account, the energy levels will not be as simple as above. However, for the theory to be supersymmetric, the energy levels must be describable as follows:

- All bosonic states have energies given by the eigenvalues of a “bosonic Hamiltonian” H_b . All fermionic states have energies given by the eigenvalues of a “fermionic Hamiltonian” H_f . These eigenvalues need not be evenly spaced.
- In a supersymmetric theory, all the eigenvalues of H_b and H_f are the same, except that the lowest eigenvalue of H_b has no partner in the spectrum of H_f .
- The lowest eigenvalue of H_b is zero, corresponding to the state with zero bosons and zero fermions.

¹⁰Something seems to have gone wrong, in that we found two bosonic states with energy $2m$ and only one fermionic state with this energy. This reflects the fact that I have been sloppy about counting states in which the particles have spin. Given the incompleteness and lack of specificity of my description, it is difficult to get around this sloppiness. You should focus on the energy levels, and ignore counting of states. When done correctly, the number of bosonic states with a given energy is the same as the number of fermionic states with the same energy.

- There is some operator \mathcal{A} such that $H_b = \mathcal{A}^\dagger \mathcal{A}$ and $H_f = \mathcal{A} \mathcal{A}^\dagger$. When \mathcal{A} acts on a bosonic state, it turns it into the fermionic state with the same energy. When \mathcal{A}^\dagger acts on a fermionic state, it turns it into the bosonic state with the same energy. When \mathcal{A} acts on the bosonic state with zero energy, it annihilates it.

You should recognize all these facts about the zero momentum sector of a supersymmetric theory, as they are the same as those we first saw in Section 2. You should also notice that in the simple case with which we began (one boson, one fermion and no interactions) in which the energy levels *are* evenly spaced, then the particular example of supersymmetric quantum mechanics which we obtain is precisely the harmonic oscillator as described in Section 3. Quantum field theory in the absence of interactions is nothing more than a bunch of harmonic oscillators, one for each point in momentum space for each species of particle.

There is a rich mathematical structure underlying supersymmetry that we can catch a glimpse of, using only what we have learned earlier in these notes. To do this, we must “double the size of the Hilbert space”, writing a new Hamiltonian

$$H = \begin{bmatrix} H_b & 0 \\ 0 & H_f \end{bmatrix}. \quad (64)$$

In this space of matrices that have twice the dimension that they did previously, define operators

$$Q \equiv \begin{bmatrix} 0 & 0 \\ \mathcal{A} & 0 \end{bmatrix} \quad Q^\dagger \equiv \begin{bmatrix} 0 & \mathcal{A}^\dagger \\ 0 & 0 \end{bmatrix}. \quad (65)$$

It is now easy to check that

$$QQ^\dagger + Q^\dagger Q = \begin{bmatrix} \mathcal{A}^\dagger \mathcal{A} & 0 \\ 0 & \mathcal{A} \mathcal{A}^\dagger \end{bmatrix} = \begin{bmatrix} H_b & 0 \\ 0 & H_f \end{bmatrix} = H \quad (66)$$

In quantum mechanics, $AB + BA$ is called the anticommutator of the operators A and B , and is denoted by $\{A, B\}$. We therefore have demonstrated that

$$\{Q, Q^\dagger\} = H. \quad (67)$$

The operator Q is called “a supersymmetry generator”, and the relationship (67) between Q and the Hamiltonian is called the “supersymmetry algebra”. If we return from our supersymmetric quantum mechanics (namely the zero momentum sector) to the full supersymmetric quantum field theory, we would find four different Q operators, and four equations with a structure very similar to (67): the operators that occur on the right hand sides of these four equations are H , p_x , p_y and p_z , which is not surprising since these four quantities form a Lorentz four-vector. The full description of the supersymmetry algebra requires more formalism than I have introduced so far. Earlier in 8.05, you have explored the consequences

of the “algebra of rotation generators”, otherwise known as the commutation relations for the angular momentum operators, and from this algebra you have deduced much of the consequences of rotational invariance in quantum mechanics. When (in graduate school) you explore the consequences of the algebra of supersymmetry generators exemplified by (67), from this algebra you will deduce much of the consequences of supersymmetry in quantum field theory.