ASSIGNMENT #1
VECTOR ANALYSIS, DELTA FUNCTIONS, COMPLETE SETS OF
FUNCTIONS, 2ND RANK TENSORS

Reading: Griffiths, "Advertisement" and Chapter 1; Class Notes pp 10-25.
Due: Friday September 16 in class or in the 8.07 homework box by 6:00 pm

Problems

Problem 1-1: Vector Identities Involving Cross Products

In manipulating cross products, it is useful to define $\varepsilon_{ijk}$ (the Levi-Civita antisymmetric symbol) to be:

$$\varepsilon_{ijk} = \begin{cases} 
+1 & \text{if } ijk = (123, 231, 312) \\
-1 & \text{if } ijk = (213, 321, 132) \\
0 & \text{otherwise}
\end{cases} \quad (1.1.1)$$

Note that $\varepsilon_{ijk} = -\varepsilon_{ikj} = \varepsilon_{kij}$. With this definition, the $i$-th component of the cross product of two vectors $\mathbf{A}$ and $\mathbf{B}$ can be written as

$$(\mathbf{A} \times \mathbf{B})_i = \varepsilon_{ijk} A_j B_k \quad (1.1.2)$$

where we have again used the summation convention that indices repeated twice are summed over (that is, $\varepsilon_{ijk} A_j B_k = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} A_j B_k$). In the future, we will always assume that this summation convention is implied, unless explicitly stated otherwise.

(a) From the definition in (1), show that

$$\varepsilon_{ijk} \varepsilon_{inm} = \delta_{jn} \delta_{km} - \delta_{jm} \delta_{kn} \quad (1.1.3)$$

where of course there is an implied sum over the $i$ index in (1.1.3), but the indices $j$, $k$, $n$, and $m$ are free.

(b) Using (1.1.2) and (1.1.3), show that for any vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}) \quad (1.1.4)$$

(c) Using (1.1.2) and (1.1.3), show that for any vector $\mathbf{A}$,

$$\mathbf{A} \times \nabla \times \mathbf{A} = \frac{1}{2} \nabla A^2 - (\mathbf{A} \cdot \nabla) \mathbf{A} \quad (1.1.5)$$

(d) Using (1.1.2) and (1.1.3), show that for any vectors $\mathbf{A}$ and $\mathbf{B}$,
\[ \nabla \times (A \times B) = (B \cdot \nabla) A - (A \cdot \nabla) B + A (\nabla \cdot B) - B (\nabla \cdot A) \quad (1.1.6) \]

**Problem 1-2: Properties of the Rotation Matrix R**

Griffiths equation (1.31), page 11, is \[ A_i = \sum_{j=1}^{3} R_{ij} A_j \]. If we use the convention that indices repeated twice are summed over, then this can be written as \( A_i = R_{ij} A_j \).

(a) Show that the elements \((R_{ij})\) of the three-dimensional rotation matrix must satisfy the constraint \( R_{ij} R_{jk} = \delta_{jk} \) in order to preserve the length of \( A \) for all \( A \). Here \( \delta_{jn} \) is the Kronecker delta (\( \delta_{jn} = 1 \) if \( j = n \) and 0 otherwise), and we use the summation convention above.

(b) Using the constraint above, show that \( A_i = R_{ij} A_j \). Note that we can now show that \( R_{ji} R_{ki} = \delta_{jk} \) using this relation, in a manner similar to the procedure in (a) (you do not have to show this).

(c) Using the chain rule for partial differentiation and the results of (b), show that if \( f \) is scalar function of \( r \), then \( \nabla f \) transforms as a vector, i.e., show that \( \frac{\partial f}{\partial x_i} = R_{ij} \frac{\partial f}{\partial x_j} \).

**Problem 1-3: Second Rank Tensors**

(a) Show that \( \mathbf{T} \) defined in any coordinate system by \( I_{ij} = \delta_{ij} \) is a second rank tensor (i.e., show that it satisfies Griffiths, (1.31), page 11). The results of 1-2(a)\&(b) above will be helpful.

(b) If \( \mathbf{T} \) is a second rank tensor and \( \mathbf{C} \) is any vector, the dot product of \( \mathbf{C} \) with \( \mathbf{T} \) "from the left" is a vector, denoted by \( \mathbf{C} \cdot \mathbf{T} \), and is given by
\[ (\mathbf{C} \cdot \mathbf{T})_j = C_i T_{ij} \quad (1.3.1) \]

The dot product of \( \mathbf{C} \) with \( \mathbf{T} \) "from the right" is also a vector, denoted by \( \mathbf{T} \cdot \mathbf{C} \), and is given by \( (\mathbf{T} \cdot \mathbf{C})_j = T_{ji} C_i \). For arbitrary \( \mathbf{T} \), these two different ways of taking the dot product of a vector and a 2nd rank tensor result in different vectors. However, for symmetric 2nd rank tensors \( (T_{ij} = T_{ji}) \), the two vectors are the same. In practice, in 8.07 we will only deal with symmetric 2nd rank tensors, so the distinction between \( \mathbf{C} \cdot \mathbf{T} \) and \( \mathbf{T} \cdot \mathbf{C} \) is unnecessary.

Show that \( \mathbf{C} \cdot \mathbf{T} \) as defined above is a vector.

(c) Equation (2.3.3) of the class notes gives the following equation for the conservation of momentum in fluid flow
\[
\frac{\partial}{\partial t} [\rho_{\text{mass}} \mathbf{v}] + \nabla \cdot [\rho_{\text{mass}} \mathbf{v} \mathbf{v}] = [\text{volume creation rate of momentum}] \tag{1.3.2}
\]

Use the differential equation for the conservation of mass (see eq. (2.2.2) of the class notes for the integral form), assuming that mass is not being created or destroyed, to show that equation (1.3.2) above can be written as

\[
\rho_{\text{mass}} \left[ \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] \mathbf{v} = [\text{volume creation rate of momentum}] \tag{1.3.3}
\]

This is the standard momentum equation one sees in fluid mechanics, where the term on the right hand side is the force per unit volume.

(d) For any vector function \( \mathbf{A}(\mathbf{r}) \), define the second rank tensor \( \mathbf{T}(\mathbf{r}) \) by

\[
\mathbf{T} = \mathbf{A} \mathbf{A} - \frac{1}{2} \mathbf{I} A^2 \quad \text{(that is, } T_{ij} = A_i A_j - \frac{1}{2} \delta_{ij} A^2 \textrm{)} \tag{1.3.4}
\]

Show that \( \mathbf{T} \) defined in this ways is a 2nd rank tensor if \( \mathbf{A} \) is a vector.

(e) The divergence "from the left" of a 2nd rank tensor is a vector, and is defined by

\[
\left( \nabla \cdot \mathbf{T} \right)_j = \frac{\partial}{\partial x_i} T_{ij} \tag{1.3.5}
\]

where \( x_1 = x, x_2 = y, \) and \( x_3 = z \). With this definition, and using (1.3.4) above for \( \mathbf{T} \) in terms of the vector \( \mathbf{A} \), show that

\[
\nabla \cdot \mathbf{T} = \mathbf{A} (\nabla \cdot \mathbf{A}) + (\nabla \times \mathbf{A}) \times \mathbf{A} \tag{1.3.6}
\]

(you will find useful the results from Problem 1-1(c)). Again, we can also define a divergence of \( \mathbf{T} "\text{from the right}" \), in analogy with the dot product above, but if the tensor is symmetric, there is no difference in the two.

The form of the 2nd rank tensor given in equation (1.3.4) above is the form of the Maxwell Stress Tensor, where \( \mathbf{A} \) is either the electric field \( \mathbf{E} \) or the magnetic field \( \mathbf{B} \). This tensor is a fundamental tensor in electromagnetism, and is related to the momentum flux density in the electromagnetic field. The identity in equation (1.3.6) above, along with Maxwell's equations, can be used to derive the differential form of the conservation of momentum equation for particles and fields in electromagnetism.
Problem 1-4: The Dirac Delta Function

One of the most used identities in this course is be the relation

\[-\nabla^2 \frac{1}{4\pi r} = -\nabla \left[ \nabla \frac{1}{4\pi r} \right] = \nabla \cdot \left[ \frac{\hat{r}}{4\pi r^2} \right] = \delta^3 (r) = \delta(x) \delta(y) \delta(z) \] (1.4.1)

It turns out of course (see Griffiths 1.5.1, p 46) that \(-\nabla^2 \frac{1}{4\pi r}\) is zero everywhere except at the origin, and ill-defined there. To get a better feel for the fact that \(-\nabla^2 \frac{1}{4\pi r}\) is a delta function, let's look at a different function which approaches \(-\frac{1}{4\pi r}\) in some limit, but which is well behaved everywhere. The function is \(f_a(r) = -\frac{1}{4\pi} \frac{1}{\sqrt{r^2 + a^2}}\). For a non-zero, \(f_a(r)\) is well-behaved everywhere, and \(\lim_{a \to 0} f_a(r) = -\frac{1}{4\pi r}\).

(a) Calculate \(g_a(r) = \nabla^2 f_a(r)\) and show that it is also well behaved for all \(r\). Sketch \(g_a(r)\) for some value of \(a\) as a function of \(r/a\).

(b) Show that \(\int_{all \ space} g_a(r) \, d^3 x = 1\).

(c) Show that \(\lim_{a \to 0} g_a(r) = 0 \quad if \quad r \neq 0\).

Thus in the limit that \(a\) goes to zero, our well-behaved function \(g_a(r)\) exhibits the properties we expect of a three-dimensional delta function.

Problem 1-5: Constructing a complete set of orthogonal functions

In the lecture notes we defined complete sets of orthogonal functions on a finite interval, and in this problem you will explicitly construct the first four functions of a denumerably infinite set. Consider the set of polynomials \(\{P_n(x)\}_{k=0}^n\) defined on the interval \([-1,1]\), where the \(n\)th function is a polynomial of order \(n\), given by

\[P_n(x) = \sum_{l=0}^{n} a_{nl} x^l\] (1.5.1)
Assume that the coefficients $a_n$ are such that these functions are orthogonal. Assume also that these functions are normalized such that for all $n$, $P_n(1) = 1$. Using these two conditions, find expressions for the first four of these polynomials, that is for $n = 0, 1, 2, 3$.

**Problem 1-6: Using a LIC program to construct three LIC maps.**

This problem is intended to help you get some feeling for curl free functions, divergence free functions, and functions which have a non-zero divergence and curl. It requires that you use a java applet on the web. To see if your computer will run these applets, go to the following url and follow the directions there.

http://www-pw.physics.uiowa.edu/das2/javaPlatformTest.jbf.html

If even after you have done this your computer will not run the applet, go to any Athena cluster and you should be able to run the applet below. Now go to

http://web.mit.edu/viz/netbeans/jnlp_web/mappingfields.jnlp

to get to the field mapping applet. Read the directions and try out a few of the examples.

(a) Construct a LIC image of a 2D vector field that is not any of the examples given, that has zero divergence. Save a jpg image of that field, print it out, and hand it in with your homework along with the analytic function that defines it.

(b) Construct a LIC image of a 2D vector field that is not any of the examples given, that has zero curl. Save a jpg image of that field, print it out, and hand it in with your homework along with the analytic function that defines it.

(c) Construct a LIC image of a 2D vector field that is not any of the examples given, that has both non-zero curl and non-zero divergence. Save a jpg image of that field, print it out, and hand it in with your homework along with the analytic function that defines it.

If you are particularly impressed with any of the images three images you generate above, attach them to an email to jbelcher@mit.edu, and I will post them on the 8.07 web page with your name and the appropriate equations.