DETACH THESE SHEETS & USE AT YOUR CONVENIENCE

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_o}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_o \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\phi(\mathbf{r},t) = \frac{1}{\left[1-\hat{\mathbf{n}}(t'_{ret})\cdot\boldsymbol{\beta}(t'_{ret})\right]}\frac{q}{4\pi\,\varepsilon_o}\frac{1}{\left|\mathbf{r}-\mathbf{X}(t'_{ret})\right|}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mathbf{u}(t'_{ret})}{c^2}\phi(\mathbf{r},t)$$

$$t'_{ret} = t - \left|\mathbf{r}-\mathbf{X}(t'_{ret})\right|/c$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{E}(\mathbf{r},t) = \left[\frac{q}{4\pi\,\varepsilon_o}\frac{\hat{\mathbf{n}}-\boldsymbol{\beta}}{\gamma_u^2(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3 R^2}\right]_{ret} + \left[\frac{q}{4\pi\,\varepsilon_o}\frac{1}{c}\frac{\hat{\mathbf{n}}\ \mathbf{x}\left\{(\hat{\mathbf{n}}-\boldsymbol{\beta})\ \mathbf{x}\ \dot{\boldsymbol{\beta}}\right\}}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3 R}\right]_{ret}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\left[\hat{\mathbf{n}}\ \mathbf{x}\ \mathbf{E}\right]_{ret}$$

$$\frac{dW_{rad}}{d\Omega\, dt'} = \frac{q^2}{(4\pi)^2 c\varepsilon_o}\frac{\left|\hat{\mathbf{n}}\times\left[(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|^2}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^5}$$

$$\frac{dW_{rad}}{dt'} = \frac{1}{4\pi\,\varepsilon_o}\frac{2q^2}{3c}\gamma^6\left[\left|\dot{\boldsymbol{\beta}}\right|^2 - \left|\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right|^2\right]$$

$$\begin{pmatrix}\bar{x}^0\\ \bar{x}^1\\ \bar{x}^2\\ \bar{x}^3\end{pmatrix} = \begin{pmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x^0\\ x^1\\ x^2\\ x^3\end{pmatrix}$$

$$\begin{pmatrix}x^0\\ x^1\\ x^2\\ x^3\end{pmatrix} = \begin{pmatrix}\gamma & \gamma\beta & 0 & 0\\ \gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\bar{x}^0\\ \bar{x}^1\\ \bar{x}^2\\ \bar{x}^3\end{pmatrix}$$

$$J^\mu = (c\rho, J_x, J_y, J_z)$$

$$A^\mu = (\frac{\phi}{c}, A_x, A_y, A_z)$$

$$\bar{E}_x = E_x$$

$$\bar{E}_y = \gamma\left(E_y - \mathrm{v}\,B_z\right)$$

$$\bar{E}_z = \gamma\left(E_z + \mathrm{v}B_y\right)$$

$$\bar{B}_x = B_x$$

$$\bar{B}_y = \gamma\left(B_y + \frac{\mathrm{v}}{c^2}E_z\right)$$

$$\bar{B}_z = \gamma(B_z - \frac{\mathrm{v}}{c^2}E_y)$$

Contra-variant 4 vector transforms as

$$\begin{pmatrix}\bar{S}^0\\ \bar{S}^1\\ \bar{S}^2\\ \bar{S}^3\end{pmatrix} = \begin{pmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}S^0\\ S^1\\ S^2\\ S^3\end{pmatrix}$$

Co-variant 4 vector transforms as

$$\begin{pmatrix}\bar{S}_0\\ \bar{S}_1\\ \bar{S}_2\\ \bar{S}_3\end{pmatrix} = \begin{pmatrix}\gamma & \gamma\beta & 0 & 0\\ \gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}$$

$$\begin{pmatrix}S^0\\ S^1\\ S^2\\ S^3\end{pmatrix} = \begin{pmatrix}-S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = (-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

$$\partial_\mu F^{\mu\nu} = -\mu_o J^\nu \qquad \partial_\mu G^{\mu\nu} = 0$$

$$X^\mu(t) = \begin{pmatrix} c\,t \\ \mathbf{X}(t) \end{pmatrix}$$

$$\mathbf{u}(t) = \frac{d\,\mathbf{X}(t)}{dt}$$

$$d\tau = d\,t\sqrt{\left(1 - \frac{\mathbf{u}^2(t)}{c^2}\right)}$$

$$\eta^\mu = \frac{d}{d\tau} X^\mu(\tau) = \begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix}$$

$$m\frac{d}{d\tau}\eta^\mu = qF^{\mu\sigma}\eta_\sigma$$

$$m\frac{d}{d\tau}\begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix} = m\gamma_u \frac{d}{dt}\begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix}$$

$$= q\begin{pmatrix} \gamma_u \dfrac{\mathbf{E}\cdot\mathbf{u}}{c} \\ \gamma_u\left(\mathbf{E} + \mathbf{u}\,\mathbf{x}\,\mathbf{B}\right) \end{pmatrix}$$

$$Energy\ of\ \ particle = m\gamma_u c^2 = mc^2 / \sqrt{1 - \frac{u^2}{c^2}}$$

$$\mathbf{F}_{\text{radiation reaction}} = \frac{1}{4\pi\varepsilon_o}\frac{2}{3}\frac{q^2}{c^3}\frac{d^2\mathbf{V}}{dt^2}$$

Electrostatics:

$$\mathbf{E} = -\nabla\phi \qquad \phi(\mathbf{r}) = -\int_{\mathbf{r}_o}^{\mathbf{r}} \mathbf{E}(\mathbf{r}')\cdot d\,\mathbf{l}'$$

$$\phi(\mathbf{r}) = \int_{all\,space} \frac{\rho(\mathbf{r}')\,d^3x'}{4\pi\varepsilon_o\left|\mathbf{r}-\mathbf{r}'\right|}$$

$$W_N = \sum_{i=1}^{N}\sum_{j<i}\frac{q_i q_j}{4\pi\varepsilon_o\left|\mathbf{r}_i - \mathbf{r}_j\right|}$$

$$W = \int\frac{1}{2}\rho(\mathbf{r})\phi(\mathbf{r})\,d^3x = \int\frac{1}{2}\varepsilon_o E^2(\mathbf{r})d^3x$$

$$\nabla^2\phi(\mathbf{r}) = -\rho(\mathbf{r})/\varepsilon_o$$

$$\int_{\substack{\text{closed}\\\text{surface}}} \mathbf{E}\cdot\hat{\mathbf{n}}\,da = \frac{q_{\text{enclosed}}}{\varepsilon_o}$$

In general in spherical coordinates:

$$\phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^l + B_{lm}r^{-l-1}\right]Y_{lm}(\theta,\phi)$$

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{+im\phi}$$

For ϕ symmetry in spherical coordinates:

$$\phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

$$\int_{-1}^{1} P_l(x)P_m(x)dx = \frac{2}{2l+1}\delta_{lm}$$

Boundary conditions on E:

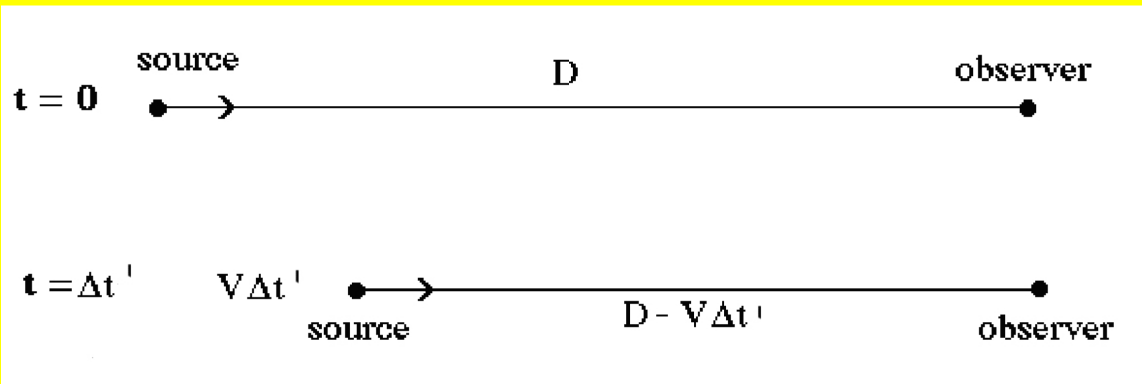
$$E_{2n} - E_{1n} = \hat{\mathbf{n}}\cdot\left(\mathbf{E}_2 - \mathbf{E}_1\right) = \sigma/\varepsilon_o$$

$$\mathbf{E}_{2t} = \mathbf{E}_{1t}$$

Problem 1: Relativistic particle motion

(a) A spaceship with speed $V < c$ moves directly toward an observer in the laboratory frame, firing laser blasts with a time separation $\Delta t'$ at the observer, where $\Delta t'$ is the time separation of the blasts as measured by our infinite grid of observers at rest in the laboratory frame. What is the time separation Δt between these laser blasts when they arrive at the observer, in terms of *V*, *c* and $\Delta t'$? Justify your answer.

Consider two laser blasts. Assume that the first blast is emitted at *t = 0*. Suppose also at time *t = 0* that the observer and the source are separated by a distance *D* (see Figure)



The observer will see the first blast arrive at a time $t_1 = D/c$. How about the second blast? Well, if it is emitted a time $\Delta t'$ after the first blast, and the source is assumed to be moving directly toward the observer at speed *V*, the source will be only a distance $D - V\Delta t'$ from the observer when the second blast is emitted. The second blast will then arrive at the observer at a time $t_2 = \Delta t' + \left(D - V\Delta t'\right)/c$ after the first blast. Since the time difference between blast as seen by the observer is $\Delta t = t_2 - t_1$, we have, as above

$$\Delta t = t_2 - t_1 = \left[\Delta t' + (D - V\Delta t')/c\right] - D/c = (1 - V/c)\Delta t'$$

We could also start with the general expression $t'_{ret} = t - \left| \mathbf{r} - \mathbf{X}(t'_{ret}) \right|/c$ and differentiate with respect to t'_{ret}, giving $1 = \frac{dt}{dt'_{ret}} + \hat{\mathbf{n}} \cdot \boldsymbol{\beta} \Rightarrow \frac{dt}{dt'_{ret}} = 1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta}$. Then applying the situation we have described above, we have $\Delta t = \left(1 - V/c\right)\Delta t'_{ret}$. I prefer the first way of doing this, since the second is mathematical and hides the physics. I am not sure how much credit I will give for this way of doing it.

(b) The four acceleration vector is defined by $\Xi^{\mu} = \frac{d}{d\tau}\eta^{\mu}$ and is given in terms of ordinary acceleration and velocity by $\Xi^{\mu} = \gamma_u^{\,2}\begin{pmatrix} \gamma_u^{\,2}\left(\dfrac{\mathbf{u}\cdot\mathbf{a}}{c}\right) \\ \mathbf{a}(t)+\gamma_u^{\,2}\mathbf{u}\left(\dfrac{\mathbf{u}\cdot\mathbf{a}}{c^2}\right)\end{pmatrix}$. From this expression for Ξ^{μ}, how would you compute the acceleration of the particle in its instantaneous rest frame given quantities measured in the lab frame? This is not a question about Lorentz transformations, it is a question about Lorentz invariants. You do not have to carry out the computation, just indicate how you would do it.

The Lorentz invariant length of the four acceleration is $\Xi^{\mu}\Xi_{\mu} = -\left(\Xi^0\right)^2+\left(\Xi^1\right)^2+\left(\Xi^2\right)^2+\left(\Xi^3\right)^2$

We can compute this given the ordinary velocity and acceleration as measured in the laboratory frame, using the expression above. But in the instantaneous rest frame of the particle, the particle speed is zero and its acceleration is $\mathbf{a}_{\text{rest fame}}$, so the invariant length of the four acceleration vector is just $(a_{\text{rest fame}})^2$. Thus we have

$$\left(a_{rest\ frame}\right)^2 = \gamma_u^{\,4}\left[-\left[\gamma_u^{\,2}\left(\frac{\mathbf{u}\cdot\mathbf{a}}{c}\right)\right]^2+\left(\mathbf{a}(t)+\gamma_u^{\,2}\mathbf{u}\left(\frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\right)\right)\cdot\left(\mathbf{a}(t)+\gamma_u^{\,2}\mathbf{u}\left(\frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\right)\right)\right]$$

You do not need to go any further than this.

Problem 2: Dynamics of Magnetic Monopoles

Let the mass of a magnetic monopole be m_m, its 4 vector position be $X^{\mu}(t) = \begin{pmatrix} ct \\ \mathbf{X}(t)\end{pmatrix}$, its ordinary velocity be $\mathbf{u}(t) = \dfrac{d\,\mathbf{X}(t)}{dt}$, and its 4 vector velocity be $\eta^{\mu} = \dfrac{d}{d\tau}X^{\mu}(\tau)$. We have for an electric monopole that $m\dfrac{d}{d\tau}\eta^{\mu} = qF^{\mu\sigma}\eta_{\sigma}$.

(a) What manifestly covariant equation do you think would be appropriate for a magnetic monopole, assuming that the non-relativistic form is $m_m\mathbf{a} = q_m\mathbf{B}$.

$$m_m\frac{d}{d\tau}\eta^{\mu} = \frac{1}{c}q_m G^{\mu\sigma}\eta_{\sigma}$$

(b) Write your manifestly covariant equation from (a) in ordinary coordinate form, that is in terms of the ordinary position and velocity of the monopole, with any time derivatives in terms of coordinate time *t*. Give both the time and space components of your 4 vector equation in terms of these quantities. Your final equations should be relativistically correct, even though written in in terms of the ordinary position and velocity of the monopole and differentials with respect to coordinate time *t*.

$$m_m \frac{d}{d\tau}\eta^\mu = \frac{1}{c} q_m G^{\mu\sigma}\eta_\sigma \quad \text{and} \quad \eta^\mu = \frac{d}{d\tau}X^\mu(\tau) = \begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix}$$

$$m_m \frac{d}{d\tau}\begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix} = \frac{1}{c} q_m \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix} \begin{pmatrix} -\gamma_u c \\ \gamma_u u_x(t) \\ \gamma_u u_y(t) \\ \gamma_u u_z(t) \end{pmatrix}$$

$$m_m \frac{d}{d\tau}\begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix} = q_m \begin{pmatrix} \gamma_u \mathbf{u}\cdot\mathbf{B}/c \\ \gamma_u \left[B_x - \frac{1}{c^2}\left(u_y E_z - u_z E_y\right)\right] \\ \gamma_u \left[B_y - \frac{1}{c^2}\left(u_z E_x - u_x E_z\right)\right] \\ \gamma_u \left[B_z - \frac{1}{c^2}\left(u_x E_y - u_y E_x\right)\right] \end{pmatrix} = q_m \begin{pmatrix} \gamma_u \mathbf{u}\cdot\mathbf{B}/c \\ \gamma_u \left[\mathbf{B} - \frac{1}{c^2}\mathbf{u}\times\mathbf{E}\right] \end{pmatrix}$$

So our equations for the magnetic monopole are

$$m_m \gamma_u \frac{d}{dt}\begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix} = q_m \begin{pmatrix} \gamma_u \mathbf{u}\cdot\mathbf{B}/c \\ \gamma_u \left[\mathbf{B} - \frac{1}{c^2}\mathbf{u}\times\mathbf{E}\right] \end{pmatrix}$$

Or

$$\frac{d}{dt} m_m \gamma_u c^2 = q_m \mathbf{u}\cdot\mathbf{B}$$

$$m_m \frac{d}{dt}\gamma_u \mathbf{u}(t) = q_m \left[\mathbf{B} - \frac{1}{c^2}\mathbf{u}\times\mathbf{E}\right]$$

Problem 3: Electrostatics

Suppose we have a sphere or radius R which has a surface charge density on its surface that is equal to $\sigma_o \cos\theta$, where θ is the spherical polar angle θ. We are going to guess that the electric field due to this surface charge density is of the form

$$\mathbf{E} = \begin{cases} \dfrac{2p\cos\theta}{4\pi\varepsilon_o r^3}\hat{\mathbf{r}} \\ +\dfrac{p\sin\theta}{4\pi\varepsilon_o r^3}\hat{\boldsymbol{\theta}} \quad r > R \\ -E_o\hat{\mathbf{z}} \quad r < R \end{cases} \tag{0.1}$$

Using the boundary condition on **E**, find the value of E_o and p in terms of R and σ_o and fundamental constants. You must justify your answer to get credit, not simply state the results.

To determine the constants we have at the equator that the tangential component of **E** is continuous, so we have

$$E_o = \frac{1}{4\pi\varepsilon_o}\frac{p}{R^3}$$

At the poles, we have the normal component of E jumping by σ_o, so

$$\frac{2p}{4\pi\varepsilon_o R^3} + E_o = \frac{\sigma_o}{\varepsilon_o}$$

Solving these equations gives

$$E_o = \frac{\sigma}{3\varepsilon_o} \qquad\qquad p = \frac{4\pi R^3}{3}\sigma_o$$