

DETACH THESE SHEETS & USE AT YOUR CONVENIENCE

Maxwell in differential form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

Maxwell in integral form:

$$\int_{\text{closed surface}} \mathbf{E} \cdot \hat{\mathbf{n}} da = \frac{q_{\text{enclosed}}}{\epsilon_0}$$

$$\int_{\text{bounding contour}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_{\text{open surface}} \mathbf{B} \cdot \hat{\mathbf{n}} da$$

$$\int_{\text{bounding contour}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \left[I_{\text{through}} + \epsilon_0 \frac{d}{dt} \int_{\text{open surface}} \mathbf{E} \cdot \hat{\mathbf{n}} da \right]$$

$$\int_{\text{closed surface}} \mathbf{B} \cdot \hat{\mathbf{n}} da = 0$$

$$C = \frac{Q}{|\Delta V|} \quad \Delta V = -\int_a^b \mathbf{E} \cdot d\vec{\mathbf{l}}$$

$$\text{Energy} = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} C V^2$$

$$\Delta V = IR$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \mathbf{A} = -\mu_0 \mathbf{J}$$

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \phi = -\frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$$

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(\mathbf{r}', t'_{\text{ret}})}{|\mathbf{r} - \mathbf{r}'|} d^3x'$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t'_{\text{ret}})}{|\mathbf{r} - \mathbf{r}'|} d^3x'$$

$$t'_{\text{ret}} = t - |\mathbf{r} - \mathbf{r}'| / c$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{r^3} \frac{[3\hat{\mathbf{n}}(\mathbf{p} \cdot \hat{\mathbf{n}}) - \mathbf{p}]}{4\pi \epsilon_0}$$

$$+ \frac{1}{c r^2} \frac{[3\hat{\mathbf{n}}(\dot{\mathbf{p}} \cdot \hat{\mathbf{n}}) - \dot{\mathbf{p}}]}{4\pi \epsilon_0}$$

$$+ \frac{1}{rc^2} \frac{(\ddot{\mathbf{p}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}}{4\pi \epsilon_0}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \left[\frac{\dot{\mathbf{p}}}{r^2} + \frac{\ddot{\mathbf{p}}}{c r} \right] \times \hat{\mathbf{n}}$$

all evaluated at $t' = t - r/c$

$$\frac{\partial}{\partial t} \left[\frac{1}{2} \epsilon_0 E^2 + \frac{B^2}{2\mu_0} \right] + \nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) = -\mathbf{E} \cdot \mathbf{J}$$

$$\text{Energy density of field} = \frac{1}{2} \epsilon_0 E^2 + \frac{B^2}{2\mu_0}$$

$$\text{Energy flux density} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}$$

If $\vec{\mathbf{T}}$ is a second rank tensor, then $\mathbf{C} \cdot \vec{\mathbf{T}}$ and $\vec{\mathbf{T}} \cdot \mathbf{C}$ are both vectors, where

$$(\mathbf{C} \cdot \vec{\mathbf{T}})_i = C_j T_{ji}$$

$$(\vec{\mathbf{T}} \cdot \mathbf{C})_i = C_j T_{ij}$$

$$\frac{\partial}{\partial t} [\epsilon_0 \mathbf{E} \times \mathbf{B}] + \nabla \cdot (-\vec{\mathbf{T}}) = -[\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}]$$

$$\text{Momentum density} = \epsilon_0 \mathbf{E} \times \mathbf{B}$$

$$\text{Momentum flux density} = -\vec{\mathbf{T}}$$

$$\frac{\partial}{\partial t} [\epsilon_o \mathbf{r} \times (\mathbf{E} \times \mathbf{B})] + \nabla \cdot (-\mathbf{r} \times \ddot{\mathbf{T}}) = -\mathbf{r} \times [\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}] \quad \mathbf{m}(t') = \frac{1}{2} \int \mathbf{r}' \times \mathbf{J}(\mathbf{r}', t') d^3 x' = \frac{q}{2} \mathbf{X}(t') \times \mathbf{u}(t')$$

$$\text{Angular momentum density} = \epsilon_o \mathbf{r} \times (\mathbf{E} \times \mathbf{B})$$

$$\text{Angular momentum flux density} = -\mathbf{r} \times \ddot{\mathbf{T}}$$

$$\ddot{\mathbf{T}} = \epsilon_o \left[\mathbf{E} \mathbf{E} - \frac{1}{2} \ddot{\mathbf{I}} E^2 \right] + \frac{1}{\mu_o} \left[\mathbf{B} \mathbf{B} - \frac{1}{2} \ddot{\mathbf{I}} B^2 \right]$$

$$\ddot{\mathbf{T}}_E \cdot \hat{\mathbf{n}} \text{ lies in the plane defined by } \mathbf{E} \text{ and } \hat{\mathbf{n}}.$$

$$\text{The magnitude of } \ddot{\mathbf{T}}_E \cdot \hat{\mathbf{n}} \text{ is always } \frac{1}{2} \epsilon_o E^2$$

If you go an angle θ to get to $\mathbf{E}$ from $\hat{\mathbf{n}}$, go an angle 2θ in the same sense to get to $\ddot{\mathbf{T}}_E \cdot \hat{\mathbf{n}}$

All of the above apply to $\ddot{\mathbf{T}}_B \cdot \hat{\mathbf{n}}$ except for the fact that $|\ddot{\mathbf{T}}_B \cdot \hat{\mathbf{n}}|$ is always $B^2 / 2\mu_o$

$$\mathbf{p}(t') = \int \mathbf{r}' \rho(\mathbf{r}', t') d^3 x'$$

$$\frac{dW}{d\Omega dt} = \frac{1}{\mu_o} (\mathbf{E} \times \mathbf{B}) \cdot (r^2 \hat{\mathbf{r}})$$

$$\frac{dW_{el dip}}{d\Omega dt} = \frac{c r^2}{\mu_o} \left[\frac{\mu_o}{4\pi} \frac{\ddot{\mathbf{p}} \times \hat{\mathbf{n}}}{c r} \right]^2 = \frac{\mu_o \ddot{\mathbf{p}}^2}{(4\pi)^2 c} \sin^2 \theta$$

$$\frac{dW_{el dip}}{dt} = \frac{1}{4\pi \epsilon_o} \frac{2}{3} \frac{|\ddot{\mathbf{p}}|^2}{c^3}$$

$$\frac{dW_{el dip}}{dt} = \frac{1}{4\pi \epsilon_o} \frac{2}{3} \frac{q^2 a^2}{c^3}$$

$$\mathbf{m}(t') = \frac{1}{2} \int \mathbf{r}' \times \mathbf{J}(\mathbf{r}', t') d^3 x'$$

$$\frac{dW_{mag dip}}{dt} = \frac{\mu_o}{4\pi} \frac{2|\ddot{\mathbf{m}}|^2}{3c^3}$$

$$\rho(\mathbf{r}, t) = q \delta^3(\mathbf{r} - \mathbf{X}(t))$$

$$\mathbf{J}(\mathbf{r}, t) = q \mathbf{u}(t) \delta^3(\mathbf{r} - \mathbf{X}(t))$$

$$\mathbf{u}(t) = \frac{d}{dt} \mathbf{X}(t)$$

$$\mathbf{p}(t') = \int \mathbf{r}' \rho(\mathbf{r}', t') d^3 x' = q \mathbf{X}(t')$$

For a localized distribution of current,

$$\int_{\text{all space}} \mathbf{J}(\mathbf{r}', t') d^3 x' = - \int_{\text{all space}} (\mathbf{r}' [\nabla' \cdot \mathbf{J}(\mathbf{r}', t')]) d^3 x'$$

$$\frac{\partial \rho(\mathbf{r}', t')}{\partial t'} + \nabla' \cdot \mathbf{J}(\mathbf{r}', t') = 0$$

$$|\mathbf{r} - \mathbf{r}'| = r - \hat{\mathbf{n}} \cdot \mathbf{r}' + \dots \quad \hat{\mathbf{n}} = \mathbf{r} / |\mathbf{r}|$$

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} + \frac{\hat{\mathbf{n}} \cdot \mathbf{r}'}{r^2} + \dots$$

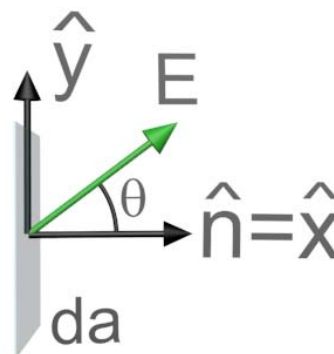
$$E_{\text{radiation}} = c B_{\text{radiation}}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = 2 \cos^2 \theta - 1$$

Problem 1

We have an imaginary vector area $d\vec{a} = da \hat{n} = da \hat{x}$, as shown in the sketch. The electric field $\vec{E}$ makes an angle θ with respect to the $\hat{n} = \hat{x}$ direction, and the $\hat{y}$ axis is in the plane formed by $\vec{E}$ and $\hat{n}$, as indicated.

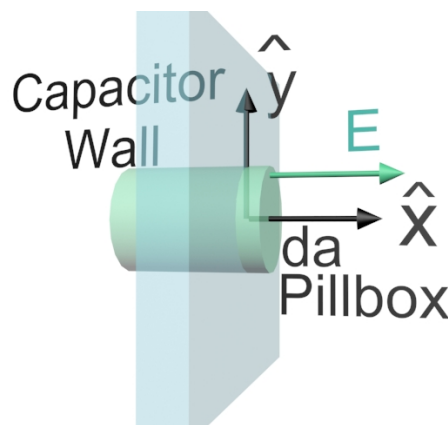


a. If the second rank tensor $\vec{T} = \epsilon_0 \left[\vec{E}\vec{E} - \frac{1}{2} \vec{I} E^2 \right]$, what are

the x, y, and z components of vector $\vec{T} \cdot \hat{n} = \vec{T} \cdot \hat{x}$ in terms of the quantities given? You must **derive** your answers from the definition of $\vec{T}$ and $\vec{T} \cdot \hat{n}$ and **show** the derivation, to get credit.

$$\begin{aligned} \vec{E} &= E \cos \theta \hat{x} + E \sin \theta \hat{y} & (\vec{T} \cdot \hat{n})_i &= T_{ij} n_j = T_{ix} \\ (\vec{T} \cdot \hat{n}) &= T_{xx} \hat{x} + T_{yx} \hat{y} + T_{zx} \hat{z} = \epsilon_0 \left(E_x E_x - \frac{1}{2} E^2 \right) \hat{x} + \epsilon_0 E_x E_y \hat{y} \\ \vec{T} \cdot \hat{n} &= \epsilon_0 E^2 \left(\cos^2 \theta - \frac{1}{2} \right) \hat{x} + \epsilon_0 E^2 \sin \theta \cos \theta \hat{y} \\ &= \frac{1}{2} \epsilon_0 E^2 (\cos 2\theta \hat{x} + \sin 2\theta \hat{y}) \end{aligned}$$

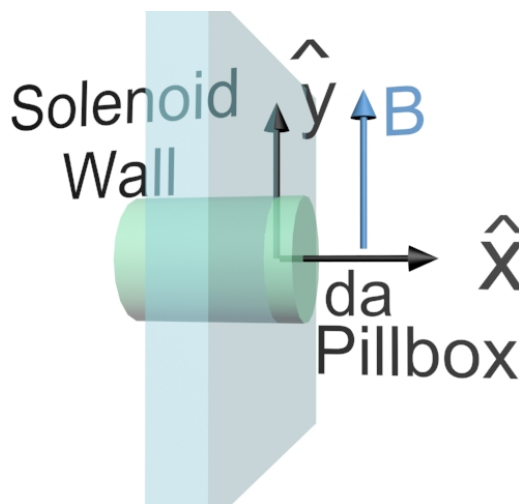
b. One side of a capacitor plate is shown in the figure, along with an imaginary pill-box enclosing the plate, with the right end of the pill box extending into the space between the capacitor plates. The electric field strength between the plates is E with the orientation shown. What is the value of $\oint \vec{T} \cdot \hat{n} da$ integrated over the pillbox. The area of the top and bottom of the pillbox is da . The electric field is zero in the region to the left of the capacitor wall shown. Justify your answer.



From the above, we see that the only contribution to the integral $\oint \vec{T} \cdot \hat{n} da$ is from the right face (the integral over the sides of the cylinder sum to zero when we do the complete integral). The integral over the right face represents a “pull”, and we have

$$\oint \vec{T} \cdot \hat{n} da = \frac{1}{2} \epsilon_0 E^2 da \hat{x}$$

c. One side of a solenoid is shown in the figure, along with an imaginary pill-box enclosing the solenoid wall, with the right end of the pill box extending into the space interior to the solenoid. The magnetic field strength in the interior of the solenoid is B , with the orientation shown. What is the value of $\oint \vec{T} \cdot \hat{n} da$ integrated over the pillbox? The area of the top and bottom of the pillbox is da . The magnetic field is zero in the region to the left of the solenoid wall shown. Justify your answer.



From the above, we see that the only contribution to the integral $\oint \vec{T} \cdot \hat{n} da$ is from the right face (the integral over the sides of the cylinder sum to zero when we do the complete integral). The integral over the right face represents a “push”, and we have

$$\oint \vec{T} \cdot \hat{n} da = -\frac{B^2}{2\mu_0} da \hat{x}$$

Problem 2:

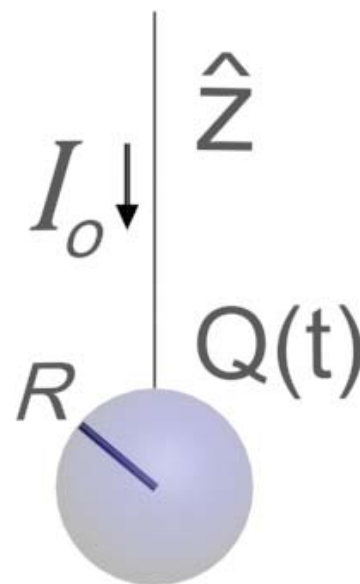
A metal sphere of radius R whose center is at the origin is being charged by a constant current I_o carried by a wire extending from the surface of the sphere vertically upwards to infinity along the z -axis (see sketch). The current in the wire is flowing downward in the $-\hat{z}$ direction, and the current density $\mathbf{J}_{\text{wire}}$ is given by

$$\mathbf{J}_{\text{wire}} = -I_o \delta(x) \delta(y) \hat{z} \quad \text{for } z \geq R$$

We have $dQ/dt = I_o \rightarrow Q(t) = I_o t$.

Once the charge carried by the wire arrives at the surface of the sphere, we assume that it spreads out uniformly over the surface of the sphere, carried by currents on the sphere which are **always perpendicular** to the radial direction.

We also assume that the electric field for this situation is just the quasi-static field, that is, we assume that



$$\mathbf{E}(\mathbf{r}, t) = \begin{cases} \frac{Q(t)}{4\pi\epsilon_0 r^2} \hat{r} & r \geq R \\ 0 & r < R \end{cases}$$

- a. What is the total electric energy stored at time t , e.g. what is $\int_{\text{all space}} \frac{1}{2} \epsilon_0 E^2 d^3x$, in terms of the quantities given?

$$\int_{\text{all space}} \frac{1}{2} \epsilon_0 E^2 d^3x = \int d\Omega \int_R^\infty r^2 dr \frac{1}{2} \epsilon_0 \left(\frac{Q(t)}{4\pi \epsilon_0 r^2} \right)^2 = 4\pi \int_R^\infty dr \frac{1}{2} \epsilon_0 \frac{Q^2(t)}{(4\pi \epsilon_0)^2 r^2} = \frac{Q^2(t)}{8\pi \epsilon_0 R}$$

- b. What is the rate at which electromagnetic energy is being created over all space, that is, what is $\int_{\text{all space}} -\mathbf{J} \cdot \mathbf{E} d^3x$?

We only need to do the integral along the wire from R to infinity. There are currents flowing on the sphere but by assumption on the sphere the currents are perpendicular to the electric field, and therefore $\mathbf{J} \cdot \mathbf{E}$ is zero there. Thus

$$\int_{\text{all space}} -\mathbf{J} \cdot \mathbf{E} d^3x = + \int_{\text{all space}} I_0 \delta(x) \delta(y) \frac{Q(t)}{4\pi \epsilon_0 r^2} \cos \theta d^3x = I_0 \int_R^\infty \frac{Q(t)}{4\pi \epsilon_0 z^2} dz = \frac{Q(t)}{4\pi \epsilon_0 R} \frac{dQ(t)}{dt}$$

- c. We want to find the magnetic field just outside the wire on a small circle of radius ϵ at a distance $z > R$ up the z -axis. Find this magnetic field by using the integral form of the Ampere-Maxwell equation applied to the Amperean loop with radius ϵ shown in the figure. Why can we neglect the displacement term $\propto dE/dt$ when we find the expression for the magnetic field, if we assume that ϵ is arbitrarily small, but not zero? Show your work.

$$\int_{\text{bounding contour}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \left[I_{\text{through}} + \epsilon_0 \frac{d}{dt} \int_{\text{open surface}} \mathbf{E} \cdot \hat{\mathbf{n}} da \right]$$

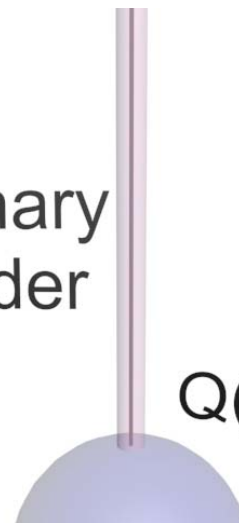
$$2\pi \epsilon B_\phi = \mu_0 \left[-I_0 + \epsilon_0 \frac{d}{dt} E \pi \epsilon^2 \right] \text{ and we can make the second term arbitrarily small, so}$$

$$B_\phi = -\frac{\mu_0 I_0}{2\pi \epsilon} \text{ where we have taken into account that the current is downwards}$$

- d. Find the total energy flow through the surface of an imaginary cylinder of radius ϵ extending from the surface of the sphere up to infinity on the z -axis (see sketch), where ϵ is arbitrarily small but not zero.

Imaginary cylinder

$Q(t)$



The energy flow is outwards and

$$\int_{\text{surface}} \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) \cdot d\vec{a} = \int_R^\infty 2\pi\epsilon dz \frac{EB}{\mu_0} = \int_R^\infty dz I_0 \frac{Q(t)}{4\pi\epsilon_0 z^2} = \frac{Q(t)}{4\pi\epsilon_0 R} \frac{dQ(t)}{dt}$$

d. How do you interpret your answers from a), b) and d) in terms of the creation, flow, and storage of electric energy? You can ignore any energy stored in the magnetic field in this interpretation. Why are you justified in ignoring that magnetic energy? Explain.

From b), we see that the external agent causing the current to flow is creating electromagnetic energy along the z-axis at the rate $\frac{Q(t)}{4\pi\epsilon_0 R} \frac{dQ(t)}{dt}$. From c) we see this electromagnetic energy

flowing out through the surface of our imaginary cylinder at the same rate at which it is being created, $\frac{Q(t)}{4\pi\epsilon_0 R} \frac{dQ(t)}{dt}$. From a) we see that the total electric energy in space is increasing at the

rate of $\frac{d}{dt} \frac{Q^2(t)}{8\pi\epsilon_0 R} = \frac{Q(t)}{4\pi\epsilon_0 R} \frac{dQ(t)}{dt}$, which equals the creation rate and the flow rate.

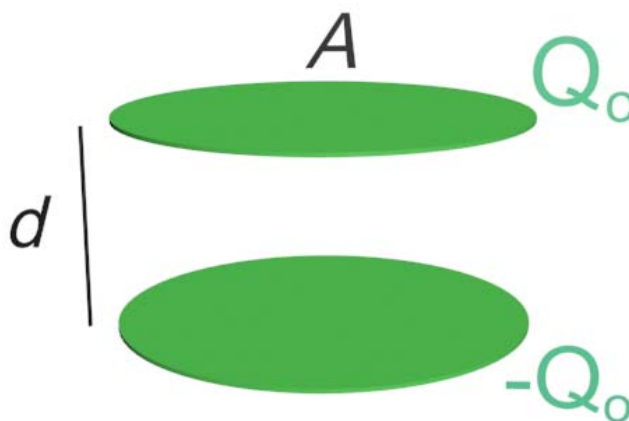
We can ignore the magnetic energy density because

$B_\phi \approx \frac{\mu_0 I_0}{2\pi r}$ even if we include the displacement current, and

$$cB_\phi / E = \frac{c\mu_0 I_0}{2\pi r} / \frac{Q}{4\pi\epsilon_0 r^2} \approx r \frac{1}{cQ} \frac{dQ}{dt} \approx \frac{r}{cT} \ll 1$$

Problem 3:

A charged capacitor consists of identical circular plates of area A separated by a distance d (see sketch). The plates initially carry charges $\pm Q_0$.



a. Assuming that the electric field is uniform between the plates of the capacitor and zero outside of the plates (that is, ignore fringing fields), show that the capacitance of this capacitor is given by $C = \epsilon_0 A / d$.

Gauss's Law applied to a Gaussian

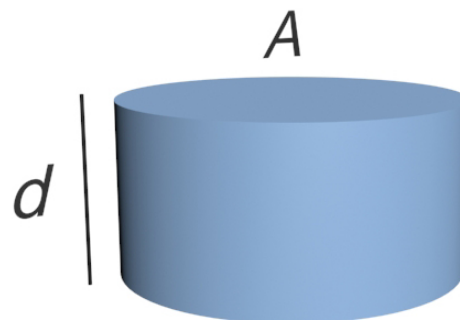
pillbox similar to the one shown in the figure in Problem 1b gives us the electric field inside the

capacitor, $daE = da \frac{Q}{A\epsilon_0} \rightarrow E = \frac{Q}{A\epsilon_0}$. The potential difference between the plates is just

$$|\Delta V| = \left| \int_a^b \mathbf{E} \cdot d\vec{l} \right| = Ed = \frac{Qd}{A\epsilon_0} \rightarrow C = \epsilon_0 A / d$$

b. We have a cylinder of resistive material, resistivity $\rho_{\text{resistivity}}$. The cylinder has area A and height d . The resistance R of this cylinder when inserted between the plates of the capacitor above is $R = d\rho_{\text{resistivity}} / A$. We insert this resistive material between the plates of the capacitor and it begins to discharge. Show that the charge on the plates will subsequently decay in time according to

$$Q = Q_0 e^{-t/RC} = Q_0 e^{-t/(\epsilon_0 \rho_{\text{resistivity}})}$$



$$\Delta V = IR = R \frac{dQ}{dt} = -\frac{Q}{C} \rightarrow Q(t) = Q_0 e^{-t/T} \quad \text{where } T = RC = \epsilon_0 \rho_{\text{resistivity}}$$

c. Consider the electrical dipole radiation from this capacitor, assuming that the electric dipole moment vector is given by $\mathbf{p} = Qd\hat{\mathbf{z}}$. An observer P sits a distance $D \gg d \approx \sqrt{A}$ from the capacitor on the x-axis, which goes through the middle of the capacitor (see sketch). Sketch on the two graphs below

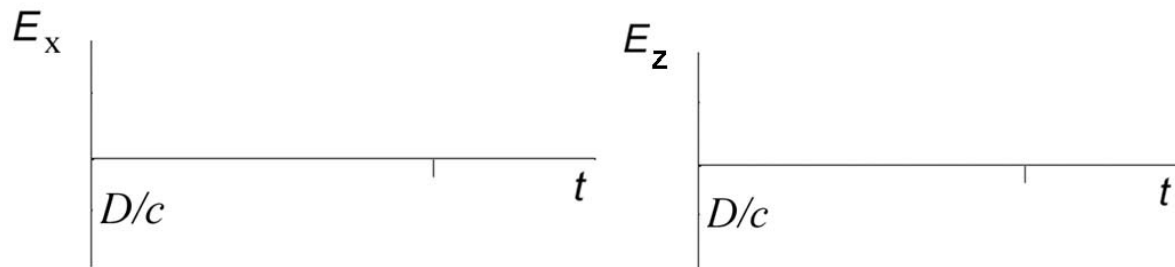


the time behavior of the x and the z components of the radiation electric field (and *only* the radiation electric field). You do not need to label the vertical axis, just get the shape right. You *do* need to label a time along the horizontal axis. Be careful of signs.

$$\ddot{\mathbf{p}} = \ddot{Q}d\hat{\mathbf{z}} = +\frac{Q_0 d e^{-t/T}}{T^2}\hat{\mathbf{z}}$$

$$\mathbf{E}_{\text{rad}}(\mathbf{r}, t) = +\frac{1}{rc^2} \frac{(\ddot{\mathbf{p}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}}{4\pi \epsilon_0} = +\frac{1}{rc^2} \frac{\left(\frac{Q_0 d e^{-t/T}}{T^2} \hat{\mathbf{z}} \times \hat{\mathbf{x}}\right) \times \hat{\mathbf{x}}}{4\pi \epsilon_0} = -\frac{1}{rc^2} \frac{Q_0 d e^{-t/T}}{4\pi \epsilon_0 T^2} \hat{\mathbf{z}}$$

So the time profile on the z-axis looks like an upside down dying exponential with an e-folding time of $T = \epsilon_0 \rho_{\text{resistivity}}$, and there is no component of the radiation field on the x-axis.



d. Compute the total energy radiated away into all solid angles in electric dipole radiation as the capacitor discharges. Find the ratio of this radiated energy to the energy originally stored in the capacitor before it began discharging (your ratio should be dimensionless). What condition do you have to put on d and A for the energy radiated away to be very small compared to the original electrostatic energy stored?

$$\frac{dW_{el\ dip}}{dt} = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{|\ddot{\mathbf{p}}|^2}{c^3} \rightarrow W_{el\ dip} = \int_0^\infty \frac{1}{4\pi\epsilon_0} \frac{2}{3c^3} \frac{Q_0^2 d^2 e^{-2t/T}}{T^4} dt = \frac{1}{4\pi\epsilon_0} \frac{T}{2} \frac{2}{3c^3} \frac{Q_0^2 d^2}{T^4}$$

$$\frac{W_{el\ dip}}{\frac{1}{2} \frac{Q_0^2}{C}} = \frac{1}{4\pi\epsilon_0} \frac{2}{3c^3} \frac{d^2}{T^3} \left(\frac{\epsilon_0 A}{d} \right) = \frac{1}{4\pi} \frac{2}{3} \frac{dA}{(cT)^3}$$

So to guarantee that the radiated energy is small compared to the stored energy, we need

$$\frac{1}{4\pi} \frac{2}{3} \frac{dA}{(cT)^3} \ll 1, \text{ which is basically the electric dipole approximation.}$$