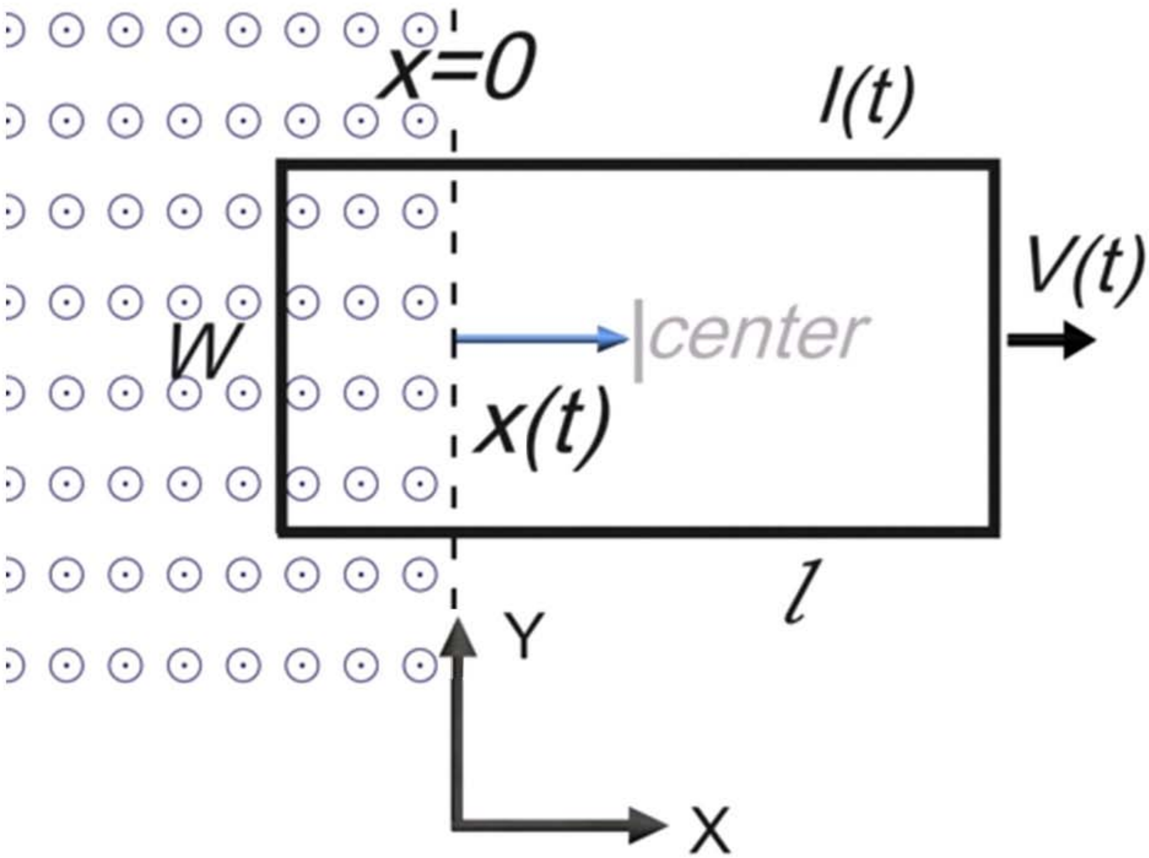
Solutions 8.07 Final Exam December 14, 2010

Problem 1: Faraday's Law



A rectangular loop lies in the *xy* plane and is made out of wire with zero resistance. The loop has mass *m*, inductance *L*, width *W*, and length *l*. It oscillates back and forth with frequency ω in the *x*-direction in an external magnetic field that is out of the page for $x < 0$ with magnitude B_o, and zero for $x > 0$. The position $x(t)$ of the center of the loop at time *t* is given by

$$x(t) = x_o \sin(\omega t) \qquad\qquad x_o < \frac{l}{2} \tag{1.1}$$

(see figure). Thus some part of the loop is always threaded by the external magnetic field. The velocity of the loop is $V(t) = dx(t)/dt$ The current in the loop is $I(t)$. At $t = 0$, the current in the loop is zero and the center of the loop is at $x = 0$.

(a) State two conservations laws for this system. Name them and give a quantitative expression for each in terms of $I(t), x(t), V(t)$ and the parameters given above. Evaluate the two constants of the motion associated with these conservation laws at *t = 0* in terms of ω, x_o, W, l, B_o and m. (Hint: if you have trouble formulating the conservation laws, you can derive them from Faraday's Law and **F** = *M***a**, but you do not have to do that if you do not need to, it is enough to simply state what is conserved).

Conservation of magnetic flux, assuming I is positive counterclockwise:

$$LI(t) + B_o W\left[\frac{1}{2}l - x(t)\right] = \frac{1}{2}B_o W\, l$$

Conservation of energy

$$\frac{1}{2}MV^2 + \frac{1}{2}LI^2 = \text{constant} = \frac{1}{2}M\left[\omega x_o\right]^2$$

(b) Use your results in (a) to deduce the frequency ω of the oscillations in terms of the parameters given above, i.e. *m*, *L*, *W*, *l*, and B_o (your answer might not contain all of these parameters).

Solving the equation for conservation of flux gives $I(t) = \dfrac{B_o W\, x(t)}{L}$

Inserting this into conservation of energy gives

$$\frac{1}{2}m\left[\omega x_o \cos(\omega t)\right]^2 + \frac{1}{2}L\left[\frac{B_o W\, x_o \sin(\omega t)}{L}\right]^2 = \frac{1}{2}m\left[\omega x_o\right]^2$$

For this equation to be true we must have

$$\frac{1}{2}m\left[\omega x_o\right]^2 = \frac{1}{2}L\left[\frac{B_o W\, x_o}{L}\right]^2$$

or $\omega = \dfrac{B_o W}{\sqrt{mL}}$

Another way to do this is as follows. If the student does it this way and works backwards to get conservation of energy and flux, I would give him or her full credit.

Using Faraday's Law and Ohm's Law, we can write

$$IR = -L\frac{dI}{dt} - \frac{d}{dt}\int_{\text{area loop}} \vec{\mathbf{B}}_{external}\cdot d\vec{\mathbf{A}},\ \text{and since } R = 0,$$

$$-L\frac{dI}{dt} - \frac{d}{dt}B_o W\left(\frac{1}{2}l - x(t)\right) = -L\frac{dI}{dt} + B_o W V(t) = 0,\ \text{so}$$

$$V(t) = \frac{L}{B_o W}\frac{dI}{dt}$$

If we look at the equation of motion of the loop it is

$$m\frac{dV}{dt} = -B_o W I \quad \text{so we have a differential equation for I which is}$$

$$m\frac{L}{B_o W}\frac{d^2 I}{dt^2} + B_o W I = 0 \text{ and therefore } \omega = \frac{B_o W}{\sqrt{mL}}$$

Problem 2: Amending Maxwell's Equations to Include Magnetic Monopoles

We want to amend Maxwell's Equations to account for the possible existence of magnetic monopoles. Let the mass of a magnetic monopole be m_m, its 4 vector position be $X^\mu(t) = \begin{pmatrix} ct \\ \mathbf{X}(t) \end{pmatrix}$, its ordinary velocity be $\mathbf{u}(t) = \frac{d\,\mathbf{X}(t)}{dt}$, and its 4 vector velocity be $\eta^\mu = \frac{d}{d\tau}X^\mu(\tau)$. We have for an electric monopole that $m\frac{d}{d\tau}\eta^\mu = qF^{\mu\sigma}\eta_\sigma$, so we guess that for a magnetic monopole we should have

$$m_m\frac{d}{d\tau}\eta^\mu = \frac{1}{c}q_m G^{\mu\sigma}\eta_\sigma \tag{1.2}$$

where we have chosen the factor of $\frac{1}{c}$ to give us in the non-relativistic limit $m_m\mathbf{a} = q_m\mathbf{B}$ for the space component of Eq. (2.1).

(a) What is Eq. (2.1) in ordinary coordinate form, that is in terms of the ordinary position and velocity of the monopole, with any time derivatives in terms of coordinate time *t*. Give both the time and space components of your 4 vector Eq. (2.1) in terms of these quantities. Your final equations should be relativistically correct, even though written in in terms of the ordinary position and velocity of the monopole and differentials with respect to coordinate time *t*.

$$m_m \frac{d}{d\tau}\eta^{\mu} = \frac{1}{c} q_m G^{\mu\sigma}\eta_{\sigma} \quad \text{and} \quad \eta^{\mu} = \frac{d}{d\tau} X^{\mu}(\tau) = \begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix}$$

$$m_m \frac{d}{d\tau}\begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix} = \frac{1}{c} q_m \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix} \begin{pmatrix} -\gamma_u c \\ \gamma_u u_x(t) \\ \gamma_u u_y(t) \\ \gamma_u u_z(t) \end{pmatrix}$$

$$m_m \frac{d}{d\tau}\begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix} = q_m \begin{pmatrix} \gamma_u \mathbf{u}\cdot\mathbf{B}/c \\ \gamma_u \left[B_x - \frac{1}{c^2}\left(u_y E_z - u_z E_y\right) \right] \\ \gamma_u \left[B_y - \frac{1}{c^2}\left(u_z E_x - u_x E_z\right) \right] \\ \gamma_u \left[B_z - \frac{1}{c^2}\left(u_x E_y - u_y E_x\right) \right] \end{pmatrix} = q_m \begin{pmatrix} \gamma_u \mathbf{u}\cdot\mathbf{B}/c \\ \gamma_u \frac{1}{c^2}\left[\mathbf{B} - \mathbf{u}\times\mathbf{E}\right] \end{pmatrix}$$

So our equations for the magnetic monopole are

$$m_m \gamma_u \frac{d}{dt}\begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix} = q_m \begin{pmatrix} \gamma_u \mathbf{u}\cdot\mathbf{B}/c \\ \gamma_u \frac{1}{c^2}\left[\mathbf{B} - \mathbf{u}\times\mathbf{E}\right] \end{pmatrix}$$

Or

$$\frac{d}{dt} m_m \gamma_u c^2 = q_m \mathbf{u}\cdot\mathbf{B}$$

$$m_m \frac{d}{dt}\gamma_u \mathbf{u}(t) = q_m \left[\mathbf{B} - \frac{1}{c^2}\mathbf{u}\times\mathbf{E} \right]$$

(b) Let the density of magnetic monopole charge per unit volume be ρ_m, and the current density be $\mathbf{J}_m$. What is the manifestly co-variant equation that tells us how to amend the two "sourceless" Maxwell's equations, which now have source terms in them, assuming that one of these equations is $\nabla\cdot\mathbf{B} = \mu_0 \rho_m$. Write the space part of your final four vector equation in the form $\nabla \,\mathbf{x}\, \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + ?$, where you are to identify the ? term.

We defined the four vector electric current to be $J^{\mu} = (c\rho, \mathbf{J})$ and with this definition the terms in Maxwell's equations with electric sources were given in manifestly co-variant form by $\partial_{\mu}F^{\mu\nu} = -\mu_o J^{\nu}$. This leads us to define the four vector magnetic current to be $J_m^{\mu} = (c\rho_m, \mathbf{J}_m)$ and to suggest the following form for the four vector equation we seek

$$\partial_{\mu}G^{\mu\nu} = -\frac{\mu_o}{c} J_m^{\nu} \tag{1.3}$$

where we have chosen the constant multiplying the four current density in (2.2) so that the time component reduces to $\nabla \cdot \mathbf{B} = \mu_0 \rho_m$. If we look at this equation in full, we have

$$\partial_{\mu}G^{\mu\nu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

$$\partial_{\mu}G^{\mu\nu} = \begin{pmatrix} -\nabla \cdot \mathbf{B} \\ \dfrac{1}{c}\left(\dfrac{\partial \mathbf{B}}{\partial t} + \nabla \,\mathbf{x}\, \mathbf{E}\right) \end{pmatrix} = -\frac{\mu_o}{c}\begin{pmatrix} c\rho_m \\ \mathbf{J}_m \end{pmatrix} \tag{1.4}$$

This gives us the desired equation $\nabla \cdot \mathbf{B} = \mu_0 \rho_m$ for the time component and for the space component we have

$$\frac{1}{c}\left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \,\mathbf{x}\, \mathbf{E}\right) = -\frac{\mu_o}{c}\mathbf{J}_m \quad \text{or} \quad \nabla \,\mathbf{x}\, \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mu_o \mathbf{J}_m \tag{1.5}$$

(c) Before we introduced magnetic monopoles we found that we could write our conservation of energy equation as

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon_o E^2 + \frac{B^2}{2\mu_o}\right] + \nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_o}\right) = -\mathbf{E} \cdot \mathbf{J} \tag{1.6}$$

By considering $\nabla \cdot \left(\dfrac{\mathbf{E} \times \mathbf{B}}{\mu_o}\right)$, derive the new form for this equation. Note that Eq. (4.4.1) of your class notes has a sign mistake.

$$\nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right)=\frac{1}{\mu_o}\left(\mathbf{B}\cdot\nabla\times\mathbf{E}-\mathbf{E}\cdot\nabla\times\mathbf{B}\right)=\frac{1}{\mu_o}\left(\mathbf{B}\cdot\left(-\frac{\partial\mathbf{B}}{\partial t}-\mu_o\mathbf{J}_m\right)-\mathbf{E}\cdot\left(\mu_o\mathbf{J}+\mu_o\varepsilon_o\frac{\partial\mathbf{E}}{\partial t}\right)\right)$$
$$=-\frac{1}{2\mu_o}\frac{\partial\mathbf{B}\cdot\mathbf{B}}{\partial t}-\frac{\varepsilon_o}{2}\frac{\partial\mathbf{E}\cdot\mathbf{E}}{\partial t}-\mathbf{J}\cdot\mathbf{E}-\mathbf{J}_m\cdot\mathbf{B} \quad (1.7)$$

So

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon_o E^2+\frac{B^2}{2\mu_o}\right]+\nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right)=-\mathbf{E}\cdot\mathbf{J}-\mathbf{B}\cdot\mathbf{J}_m \quad (1.8)$$

Problem 3: The field of a uniformly magnetized material

Suppose we have a sphere or radius R which has a uniform permanent magnetization **M**. We are going to guess that the magnetic field due to this magnetization **M** is of the form

$$\mathbf{B}=\begin{cases}\dfrac{\mu_o}{4\pi}\dfrac{2m\cos\theta}{r^3}\hat{\mathbf{r}} \\ +\dfrac{\mu_o}{4\pi}\dfrac{m\sin\theta}{r^3}\hat{\boldsymbol{\theta}} \quad r>R \\ B_o\hat{\mathbf{z}} \quad r<R\end{cases} \quad (1.9)$$

Find the value of B_o and m in terms of R and M and fundamental constants. You must justify your answer to get credit, not simply state the results.

The magnetization current is given by

$$\kappa_{mag}=\mathbf{M}\times\hat{\mathbf{n}}=M\,\hat{\mathbf{z}}\times\hat{\mathbf{r}}=M\sin\theta\,\hat{\boldsymbol{\phi}} \quad (1.10)$$

To determine the constants we have at the poles that the normal component of B is continuous, which is the radial part, so we have

$$B_o=\frac{\mu_o}{4\pi}\frac{2m}{R^3} \quad (1.11)$$

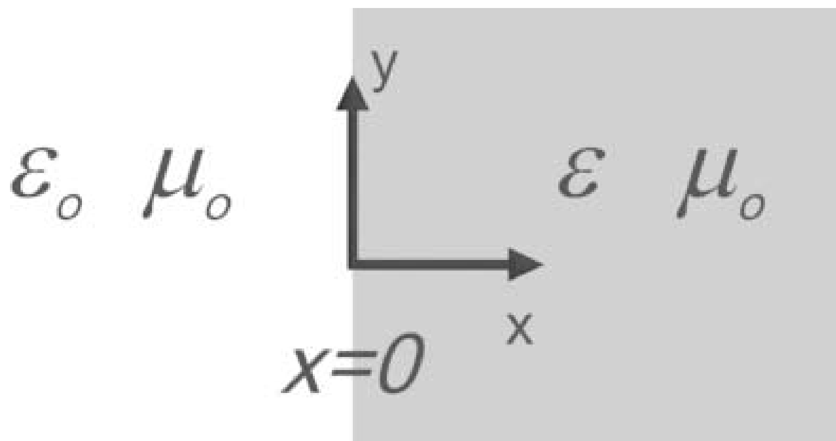
At the equator, we have the tangential component of B jumping by the surface current, so

$$\frac{\mu_o}{4\pi}\frac{m}{R^3}+B_o=\mu_o M \quad (1.12)$$

Solving these equations gives

$$B_o = \frac{2}{3}\mu_o M \qquad m_o = \frac{4\pi R^3}{3} M \tag{1.13}$$

Problem 4: Transmission and reflection at a dielectric/vacuum interface



A electromagnetic plane wave propagating from the left is incident on a dielectric with dielectric constant K_e. The incident electric field is polarized in the y direction The dielectric occupies all space for $x > 0$. For $x < 0$ we have vacuum. Our electric and magnetic fields are given by

$$\mathbf{E}(x,t) = \begin{cases} \hat{\mathbf{y}}\,\delta E_i \cos\omega\left(t - \dfrac{x}{c}\right) + \hat{\mathbf{y}}\,\delta E_r \cos\omega\left(t + \dfrac{x}{c}\right) \\ \hat{\mathbf{y}}\,\delta E_t \cos\omega\left(t - \dfrac{x\sqrt{K_e}}{c}\right) \end{cases} \tag{2.1}$$

$$\mathbf{B}(x,t) = \begin{cases} \hat{\mathbf{z}}\,\delta B_i \cos\omega\left(t - \dfrac{x}{c}\right) - \hat{\mathbf{z}}\,\delta B_r \cos\omega\left(t + \dfrac{x}{c}\right) & x < 0 \\ \hat{\mathbf{z}}\,\delta B_t \cos\omega\left(t - \dfrac{x\sqrt{K_e}}{c}\right) & x > 0 \end{cases} \tag{2.2}$$

(a) Using the boundary conditions on the electromagnetic fields at $x = 0$ and the relations between δE_i and δB_i, δE_r and δB_r, and δE_t and δB_t, find the ratio of $\delta E_t / \delta E_i$ and $\delta E_r / \delta E_i$ in terms of K_e. Verify that your answers have the right behavior when $K_e \to 1$ and $K_e \to \infty$.

We have that $\delta B_i = \delta E_i / c$, $\delta B_r = \delta E_r / c$, and $\delta B_t = \sqrt{K_e}\,\delta E_t / c$. Thus we have

$$\mathbf{E}(x,t) = \begin{cases} \hat{\mathbf{y}}\delta E_i \cos\omega\left(t - \frac{x}{c}\right) + \hat{\mathbf{y}}\delta E_r \cos\omega\left(t + \frac{x}{c}\right) \\ \hat{\mathbf{y}}\delta E_t \cos\omega\left(t - \frac{x\sqrt{K_e}}{c}\right) \end{cases} \tag{2.3}$$

$$\mathbf{B}(x,t) = \begin{cases} \hat{\mathbf{z}}(\delta E_i / c)\cos\omega\left(t - \frac{x}{c}\right) - \hat{\mathbf{z}}(\delta E_r / c)\cos\omega\left(t + \frac{x}{c}\right) & x < 0 \\ \hat{\mathbf{z}}\left(\sqrt{K_e}\delta E_t / c\right)\cos\omega\left(t - \frac{x\sqrt{K_e}}{c}\right) & x > 0 \end{cases} \tag{2.4}$$

We must have tangential **E** and tangential **B** continuous, giving

$$\delta E_i + \delta E_r = \delta E_t \qquad \delta E_i - \delta E_r = \sqrt{K_e}\delta E_t \tag{2.5}$$

These equations lead to

$$\frac{\delta E_t}{\delta E_i} = \frac{2}{1+\sqrt{K_e}} \qquad \frac{\delta E_r}{\delta E_i} = \frac{1-\sqrt{K_e}}{1+\sqrt{K_e}} \tag{2.6}$$

When $K_e \to 1$ there is no reflected wave, and the transmitted wave has the same amplitude as the incident wave. When $K_e \to \infty$ there is no transmitted wave and the amplitude of the reflected wave is the same as that of the incident wave and they are 180 degrees out of phase, given a zero electric field at $x = 0$ for any time. Both of these are the behaviors we expect.

(b) Show that rate at which energy is carried into the interface by the incident wave is equal to the sum of the rates at which energy is carried away from the interface by the reflected and transmitted wave.

We want to show that

$$\delta E_i \delta B_i = \delta E_t \delta B_t + \delta E_r \delta B_r \tag{2.7}$$

But this is

$$\delta E_i^2 = \sqrt{K_e}\delta E_t^2 + \delta E_r^2 \qquad \text{or} \qquad 1 = \sqrt{K_e}\frac{\delta E_t^2}{\delta E_i^2} + \frac{\delta E_r^2}{\delta E_i^2} \tag{2.8}$$

We have using our relations above that

$$\sqrt{K_e}\,\frac{\delta E_t^2}{\delta E_i^2}+\frac{\delta E_r^2}{\delta E_i^2}=\sqrt{K_e}\left[\frac{2}{1+\sqrt{K_e}}\right]^2+\left[\frac{1-\sqrt{K_e}}{1+\sqrt{K_e}}\right]^2=\frac{1+2\sqrt{K_e}+K_e}{\left(1+\sqrt{K_e}\right)^2}=1 \tag{2.9}$$

as we were to show.