8.07 Final Exam December 17, 2009

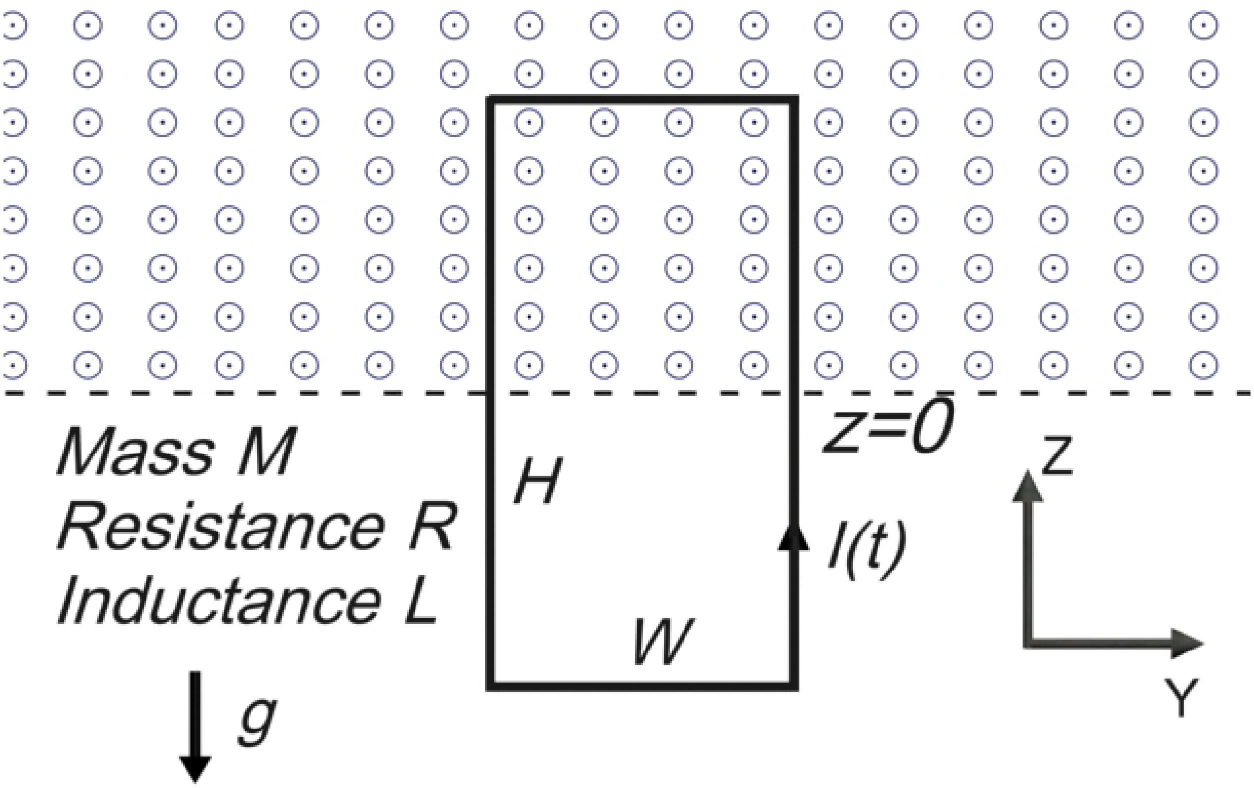
Name: ____

There are three problems. Problem 2 carries more weight.

Problem	Weight	Grade	Grader
1	30		
2	40		
3	30		
Total			

Problem 1 (30 points): Faraday's Law

A loop of mass *M*, resistance *R*, inductance *L*, height *H*, and width *W* sits in a magnetic field given by $\mathbf{B} = \hat{\mathbf{x}} \begin{cases} B_o & z \geq 0 \\ 0 & z < 0 \end{cases}$. At $t = 0$ the loop is at rest and its mid-point is at $z = 0$, as shown in the figure, ***and the current in the loop is zero at t = 0.*** The acceleration of gravity is downward at *g*. In answering all except the last question below, assume that the loop ***NEVER*** falls out of the magnetic field.

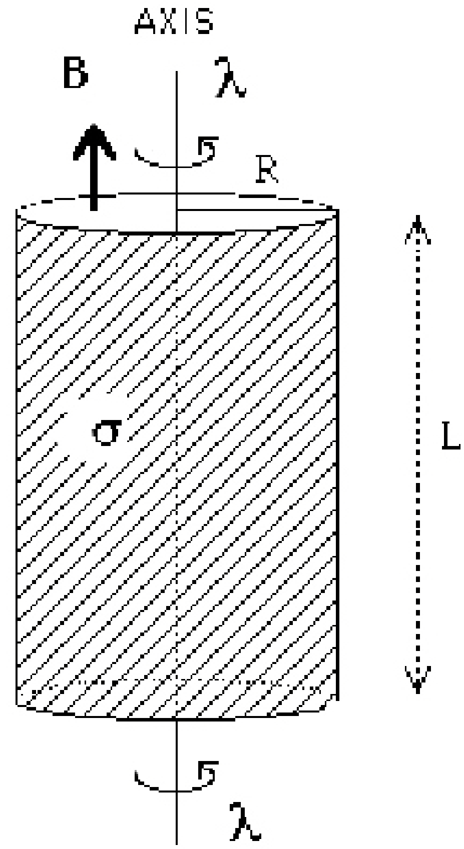


(a) What two differential equations determine the subsequent behavior of the loop? Write these two equations down in terms of the parameters given, as well as the location of the center of the loop, *z(t)*, and its speed, v*(t)=dz(t)/dt*, its acceleration, and the current in the loop at time *t*, *I(t)*, and the time derivative of that current, and the parameters given ***Take the direction of positive current to be counterclockwise, as shown in the figure.***

(b) Manipulate your equations in (a) to find an equation for the conservation of total energy.

(c) Now suppose the resistance of the loop is zero. In this situation, v*(t)* satisfies a simple second order differential equation. What is that equation? Give the solutions for v*(t)* and also *z(t)* that satisfy the boundary conditions, in terms of the parameters given. Again, assume that the loop never falls out of the magnetic field.

(d) Our assumption that the loop never falls out of the magnetic field imposes a condition on our parameters. What is that condition?



Problem 2 (40 points): Rotating Cylinder

A long cylinder has length L and radius R, with $L >> R$. The cylinder is suspended so that it can rotate freely about its axis with no friction. The cylinder carries a charge per unit area σ which is glued onto its surface, where $\sigma > 0$. Along the axis of the cylinder is a line charge with charge per unit length $\lambda = -2\pi R\sigma < 0$. Thus the electric field when the cylinder is not moving is given by

$$\mathbf{E}_{\text{static}}(r) = \begin{cases} -\dfrac{R}{r}\dfrac{\sigma}{\varepsilon_o}\hat{\mathbf{r}} & r < R \\ 0 & r > R \end{cases}$$

Now, we begin spinning the cylinder at a angular velocity $\omega(t)$ with $\omega R << c$. The motion of the charge glued onto the surface of the spinning sphere results in a surface current

$$\boldsymbol{\kappa}(t) = \sigma\,\omega(t)\,R\;\hat{\boldsymbol{\phi}}$$

We assume that we can use the quasi-static approximation to get a good approximation to the time dependent solution for **B** (good for variations in $\kappa(t)$ with time scales $T \approx \dfrac{\kappa}{d\kappa/dt} \gg \dfrac{R}{c}$)

(a) What is our quasi-static solution for **B**? Neglect fringing fields.

(b) Given this quasi-static solution for **B**, what is the electric field everywhere in space? Neglect fringing fields.

(c) What is the total magnetic energy in terms of the parameters given?

(d) Show that the total rate at which electromagnetic energy is being created as the cylinder is being spun up, $\int_{\text{all space}} -\mathbf{J}\cdot\mathbf{E}\, d^3x$, is equal to the rate at which the total magnetic energy is increasing.

(e) Calculate the flux of electromagnetic energy $\int_{surface}\left[\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right]\cdot\hat{\mathbf{r}}\,da$ through a cylindrical surface of radius r and height L for r a little greater than R and also for r a little smaller than R. Do your results agree with what you expect from (c) and (d)? Neglect fringing fields.

(f) What is the total electromagnetic angular momentum $\int \mathbf{r}\times\left[\varepsilon_o \mathbf{E}\times\mathbf{B}\right] d^3x$ in terms of the parameters given? Neglect fringing fields.

(g) Show that the total rate at which electromagnetic angular momentum is being created as the cylinder is being spun up, $\int_{\text{all space}} -\mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right] d^3x$, is equal to the rate at which the total electromagnetic angular momentum is increasing.

(h) Calculate the flux of electromagnetic angular momentum,

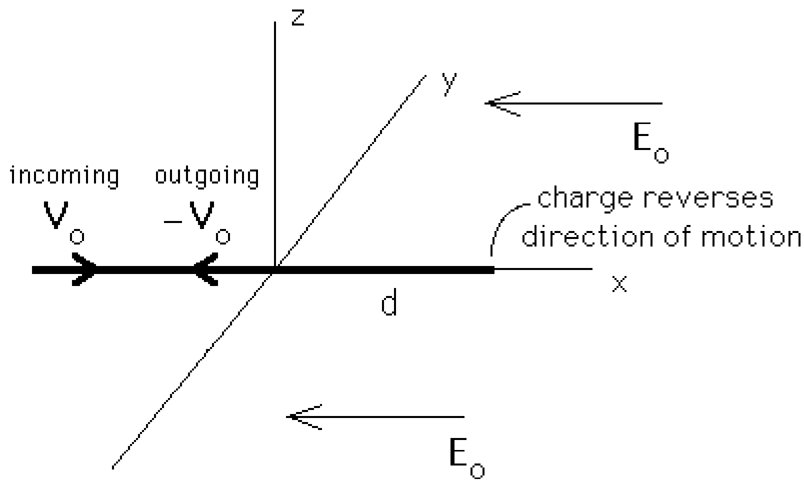
$$\int_{surface} \left[-\mathbf{r} \times \left(\overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{r}} \right) \right] da$$

through a cylinder of radius r and height L for r a little greater than R and for r a little smaller than R. Do your results agree with what you expect from (f) and (g)? As in all stress tensor calculations, figure out what components you are going to need before you calculate anything, and then just calculate those. Neglect fringing fields.

Problem 3 (30 points): Radiation by a relativistic charge

We have a static electric field $\mathbf{E}_o$ which is given by

$$\mathbf{E}_o = \begin{cases} -E_o\hat{\mathbf{x}} & x > 0 \\ 0 & x < 0 \end{cases}$$



A particle of mass *m* and charge *q* is initially located on the negative *x*-axis, with velocity $\mathbf{u}_o = \hat{\mathbf{x}}\,u_o$. *We do not assume that the speed* u_o *is small compared to c!*

(a) At *t = 0*, the particle crosses into the region *x > 0*, feels the repulsive force due to the electric field there, and begins to decelerate. It penetrates into a distance *d* along the positive *x* axis before its velocity reverses direction, and then it eventually exits the region *x > 0* , returning back down the x-axis at speed $-\mathbf{u}_o = -\hat{\mathbf{x}}\,u_o$. Find an expression for the time *T* that the particle spends in the region *x > 0*. Assume that the energy radiated is negligible, that is, ignore radiation reaction.

(b) What is the total energy radiated in this process? Give your answer in terms of q, m, E_o, u_o, and fundamental constants. The acceleration in the instantaneous rest frame of the charge in this case is particularly simple.

(c) What is the distance d in terms of the given parameters (up to this point you did not have to have d)?

(c) Let $r_c = \frac{1}{4\pi\,\varepsilon_o}\frac{q^2}{m\,c^2}$. What is the ratio of the total radiated energy from part (b) to the initial *kinetic energy* of the particle, $mc^2\left\{\frac{1}{\sqrt{1-u_o^2/c^2}}-1\right\}$? Write this ratio in terms of r_c, d, u_o and c alone. [Hint: your answer should have no dimensions--that is, it should only involve the ratios of lengths, speeds, etc.].