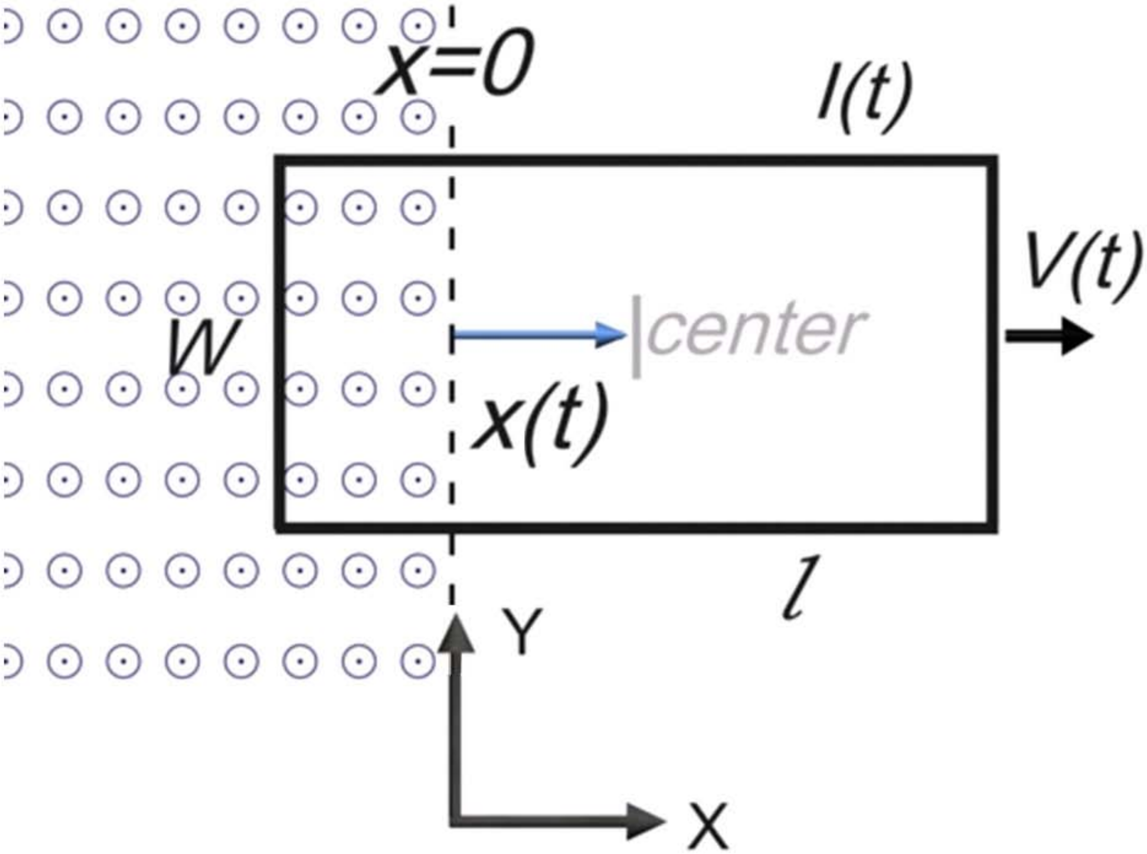
8.07 Final Exam December 14, 2010

Name: ____

There are four problems. Problem 2 carries more weight, Problem 3 carries less weight.

Problem	Weight	Grade	Grader
1	25		
2	30		
3	20		
4	25		
Total			

Problem 1: Faraday's Law



A rectangular loop lies in the xy plane and is made out of wire with zero resistance. The loop has mass m, inductance L, width W, and length l. It oscillates back and forth with frequency ω in the x-direction in an external magnetic field that is out of the page for $x < 0$ with magnitude B_o, and zero for $x > 0$. There are no frictional or gravitational forces in this problem. The position $x(t)$ of the center of the loop at time t is given by

$$x(t) = x_o \sin(\omega t) \qquad\qquad x_o < \frac{l}{2} \tag{1.1}$$

(see figure). Thus some part of the loop is always threaded by the external magnetic field. The velocity of the loop is $V(t) = dx(t)/dt$ The current in the loop is $I(t)$. At $t = 0$, the current in the loop is zero and the center of the loop is at $x = 0$.

(a) State two conservations laws for this system. Name them and give a quantitative expression for each in terms of $I(t), x(t), V(t)$ and the parameters given above. Evaluate the two constants of the motion associated with these conservation laws at $t = 0$ in terms of ω, x_o, W, l, B_o and m.

(Hint: if you have trouble formulating the conservation laws, you can derive them basic considerations, but you do not have to do that if you do not need to, it is enough to simply state what is conserved).

(b) Use your results in (a) to deduce the frequency ω of the oscillations in terms of the parameters given above, i.e. m, L, W, l, and B_o (your answer might not contain all of these parameters).

Problem 2: Amending Maxwell's Equations to Include Magnetic Monopoles

We want to amend Maxwell's Equations to account for the possible existence of magnetic monopoles. Let the mass of a magnetic monople be m_m, its 4 vector position be $X^{\mu}(t) = \begin{pmatrix} ct \\ \mathbf{X}(t) \end{pmatrix}$, its ordinary velocity be $\mathbf{u}(t) = \frac{d\,\mathbf{X}(t)}{dt}$, and its 4 vector velocity be $\eta^{\mu} = \frac{d}{d\tau} X^{\mu}(\tau)$. We have for an electric monopole that $m \frac{d}{d\tau}\eta^{\mu} = qF^{\mu\sigma}\eta_{\sigma}$, so we guess that for a magnetic monopole we should have

$$m_m \frac{d}{d\tau}\eta^{\mu} = \frac{1}{c} q_m G^{\mu\sigma}\eta_{\sigma} \tag{2.1}$$

where we have chosen the factor of $\frac{1}{c}$ to give us in the non-relativistic limit $m_m\mathbf{a} = q_m\mathbf{B}$ for the space component of Eq. (2.1).

(a) What is Eq. (2.1) in ordinary coordinate form, that is in terms of the ordinary position and velocity of the monopole, with any time derivatives in terms of coordinate time *t*. Give both the time and space components of your four vector Eq. (2.1) in terms of these quantities. Your final equations should be relativistically correct, even though written in in terms of the ordinary position and velocity of the monopole and with differentials with respect to coordinate time *t*.

(b) Let the density of magnetic monopole charge per unit volume be ρ_m, and the current density be $\mathbf{J}_m$. What is the manifestly co-variant equation that tells us how to amend the two "sourceless" Maxwell's equations, which now have source terms in them, assuming that one of these equations is $\nabla \cdot \mathbf{B} = \mu_0 \rho_m$. Write the space part of your final four vector equation in the form $\nabla \, \mathbf{x} \, \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + ?$, where you are to identify the ? term.

(c) Before we introduced magnetic monopoles we found that we could write our conservation of energy equation as

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon_o E^2 + \frac{B^2}{2\mu_o}\right] + \nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right) = -\mathbf{E}\cdot\mathbf{J} \qquad (2.2)$$

By considering $\nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right)$, derive the new form for this equation when magnetic monopoles are present. Note that Eq. (4.4.1) of your class notes has a sign mistake.

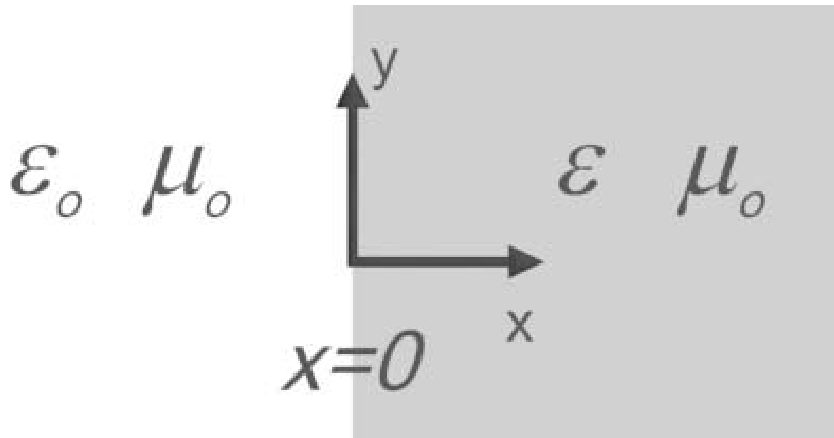
Problem 3: The magnetic field of a uniformly magnetized material

Suppose we have a sphere or radius R which has a uniform permanent magnetization $\mathbf{M}$. We are going to guess that the magnetic field due to this magnetization $\mathbf{M}$ is of the form

$$\mathbf{B} = \begin{cases} \dfrac{\mu_o}{4\pi}\dfrac{2m\cos\theta}{r^3}\hat{\mathbf{r}} \\ +\dfrac{\mu_o}{4\pi}\dfrac{m\sin\theta}{r^3}\hat{\boldsymbol{\theta}} \quad r > R \\ B_o\hat{\mathbf{z}} \quad r < R \end{cases} \tag{3.1}$$

Find the value of B_o and m in terms of R and M and fundamental constants. You must justify your answer to get credit, not simply state the results.

Problem 4: Transmission and reflection at a dielectric/vacuum interface



A electromagnetic plane wave propagating from the left is incident on a dielectric with dielectric constant K_e. The incident electric field is polarized in the y direction The dielectric occupies all space for *x > 0*. For *x < 0* we have vacuum. Our electric and magnetic fields are given by

$$\mathbf{E}(x,t) = \begin{cases} \hat{\mathbf{y}}\delta E_i \cos\omega\left(t - \dfrac{x}{c}\right) + \hat{\mathbf{y}}\delta E_r \cos\omega\left(t + \dfrac{x}{c}\right) \\ \hat{\mathbf{y}}\delta E_t \cos\omega\left(t - \dfrac{x\sqrt{K_e}}{c}\right) \end{cases} \tag{4.1}$$

$$\mathbf{B}(x,t) = \begin{cases} \hat{\mathbf{z}}\delta B_i \cos\omega\left(t - \dfrac{x}{c}\right) - \hat{\mathbf{z}}\delta B_r \cos\omega\left(t + \dfrac{x}{c}\right) & x < 0 \\ \hat{\mathbf{z}}\delta B_t \cos\omega\left(t - \dfrac{x\sqrt{K_e}}{c}\right) & x > 0 \end{cases} \tag{4.2}$$

(a) Using the boundary conditions on the electromagnetic fields at *x = 0* and the relations between δE_i and δB_i, δE_r and δB_t, and δE_t and δB_t, find the ratio of $\delta E_t / \delta E_i$ and $\delta E_r / \delta E_i$ in terms of K_e. Verify that your answers have the right behavior when $K_e \to 1$ and $K_e \to \infty$.

(b) Show that rate at which energy is carried into the interface by the incident wave is equal to the sum of the rates at which energy is carried away from the interface by the reflected and transmitted wave.