Physics 8.07 **Fall 2009**

DETACH THESE SHEETS & USE AT YOUR CONVENIENCE

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_o}$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times\mathbf{B}=\mu_o\mathbf{J}+\mu_o\varepsilon_o\frac{\partial\mathbf{E}}{\partial t}$$

$$\nabla\cdot\mathbf{B}=0$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$$\mathbf{E}=-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A}=-\mu_o\mathbf{J}$$

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi=-\frac{\rho}{\varepsilon_o}$$

$$\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}=0$$

$$\phi(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_o}\int_{all\,time}dt'\int_{all\,space}\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\times\delta(t-t'-|\mathbf{r}-\mathbf{r}'|/c)d^3x'$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_o}{4\pi}\int_{all\,time}dt'\int_{all\,space}\frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\times\delta(t-t'-|\mathbf{r}-\mathbf{r}'|/c)d^3x'$$

$$\phi(\mathbf{r},t)=\frac{1}{4\pi\,\varepsilon_o}\int\rho(\mathbf{r}',t'_{ret})\frac{d^3x'}{|\mathbf{r}-\mathbf{r}'|}$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_o}{4\pi}\int\mathbf{J}(\mathbf{r}',t'_{ret})\frac{d^3x'}{|\mathbf{r}-\mathbf{r}'|}$$

$$t'_{ret}=t-|\mathbf{r}-\mathbf{r}'|/c$$

$$\phi(\mathbf{r},t)=\frac{1}{[1-\hat{\mathbf{n}}(t'_{ret})\cdot\boldsymbol{\beta}(t'_{ret})]}\frac{q}{4\pi\varepsilon_o}\frac{1}{|\mathbf{r}-\mathbf{X}(t'_{ret})|}$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mathbf{u}(t'_{ret})}{c^2}\phi(\mathbf{r},t)$$

$$t'_{ret}=t-|\mathbf{r}-\mathbf{X}(t'_{ret})|/c$$

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{r^3}\frac{[3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}})-\mathbf{p}]}{4\pi\,\varepsilon_o}+\frac{1}{c\,r^2}\frac{[2\hat{\mathbf{n}}(\dot{\mathbf{p}}\cdot\hat{\mathbf{n}})-\dot{\mathbf{p}}]}{4\pi\,\varepsilon_o}+\frac{1}{rc^2}\frac{(\ddot{\mathbf{p}}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{4\pi\,\varepsilon_o}$$

$$\mathbf{B}(\mathbf{r},t)=\frac{\mu_o}{4\pi}\left[\frac{\dot{\mathbf{p}}}{r^2}+\frac{\ddot{\mathbf{p}}}{c\,r}\right]\times\hat{\mathbf{n}}$$

all evaluated at $t'=t-r/c$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon_o E^2+\frac{B^2}{2\mu_o}\right]+\nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right)=-\mathbf{E}\cdot\mathbf{J}$$

Energy density of field $=\frac{1}{2}\varepsilon_o E^2+\frac{B^2}{2\mu_o}$

Energy flux density = $\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}$

$$\frac{\partial}{\partial t}[\varepsilon_o\mathbf{E}\times\mathbf{B}]+\nabla\cdot\left(-\overleftrightarrow{\mathbf{T}}\right)=-[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}]$$

Momentum density = $\varepsilon_o\mathbf{E}\times\mathbf{B}$

Momentum flux density = $-\overleftrightarrow{\mathbf{T}}$

$$\frac{\partial}{\partial t}\left[\varepsilon_o\mathbf{r}\times(\mathbf{E}\times\mathbf{B})\right]+\nabla\cdot\left(-\mathbf{r}\times\overleftrightarrow{\mathbf{T}}\right)=-\mathbf{r}\times[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}]$$

Angular momentum density = $\varepsilon_o\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$

Angular momentum flux density = $-\mathbf{r}\times\overleftrightarrow{\mathbf{T}}$

$$\overleftrightarrow{\mathbf{T}}=\varepsilon_o\left[\mathbf{EE}-\frac{1}{2}\overleftrightarrow{\mathbf{I}}E^2\right]+\frac{1}{\mu_o}\left[\mathbf{BB}-\frac{1}{2}\overleftrightarrow{\mathbf{I}}B^2\right]$$

$\overleftrightarrow{\mathbf{T}}_{\mathbf{E}}\cdot\hat{\mathbf{n}}$ lies in the plane defined by $\mathbf{E}$ and $\hat{\mathbf{n}}$.

The magnitude of $\overleftrightarrow{\mathbf{T}}_{\mathbf{E}}\cdot\hat{\mathbf{n}}$ is always $\frac{1}{2}\varepsilon_o E^2$

If you go an angle θ to get to $\mathbf{E}$ from $\hat{\mathbf{n}}$, go an angle 2θ to get to $\overleftrightarrow{\mathbf{T}}_{\mathbf{E}}\cdot\hat{\mathbf{n}}$, in the same sense.

All of the above apply to $\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\cdot\hat{\mathbf{n}}$ except for the fact that $\left|\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\cdot\hat{\mathbf{n}}\right|$ is always $B^2/2\mu_o$

$$\frac{dW}{d\Omega\ dt} = \frac{c}{\mu_o}(\mathbf{E}\times\mathbf{B})\cdot(r^2\ \hat{\mathbf{r}})$$

$$\frac{dW_{el\ dip}}{d\Omega\, dt} = \frac{c\,r^2}{\mu_o}\left[\frac{\mu_o}{4\pi}\frac{\ddot{\mathbf{p}}\ \mathbf{x}\ \hat{\mathbf{n}}}{c\ r}\right]^2 = \frac{\mu_o\ \ddot{p}^{\,2}}{(4\pi)^2 c}\sin^2\theta$$

$$\frac{dW_{el\ dip}}{dt} = \frac{1}{4\pi\,\varepsilon_o}\frac{2}{3}\frac{|\ddot{\mathbf{p}}|^2}{c^3}$$

$$\frac{dW_{el\ dip}}{dt} = \frac{1}{4\pi\,\varepsilon_o}\frac{2}{3}\frac{q^2 a^2}{c^3}$$

$$\frac{dW_{mag\ dip}}{dt} = \frac{\mu_o}{4\pi}\frac{2|\ddot{\mathbf{m}}|^2}{3c^3}$$

$$\mathbf{E}(\mathbf{r},t) = \left[\frac{q}{4\pi\,\varepsilon_o}\frac{\hat{\mathbf{n}}-\boldsymbol{\beta}}{\gamma_u^2(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3 R^2}\right]_{ret} + \left[\frac{q}{4\pi\,\varepsilon_o}\frac{1}{c}\frac{\hat{\mathbf{n}}\ \mathbf{x}\left\{(\hat{\mathbf{n}}-\boldsymbol{\beta})\ \mathbf{x}\ \dot{\boldsymbol{\beta}}\right\}}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3 R}\right]_{ret}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\left[\hat{\mathbf{n}}\ \mathbf{x}\ \mathbf{E}\right]_{ret}$$

$$\frac{dW_{rad}}{d\Omega\, dt'} = \frac{q^2}{(4\pi)^2 c\varepsilon_o}\frac{\left|\hat{\mathbf{n}}\times\left[(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|^2}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^5}$$

$$\frac{dW_{rad}}{dt'} = \frac{1}{4\pi\,\varepsilon_o}\frac{2q^2}{3c}\gamma^6\left[\left|\dot{\boldsymbol{\beta}}\right|^2 - \left|\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right|^2\right]$$

$$\begin{pmatrix}\bar{x}^0\\ \bar{x}^1\\ \bar{x}^2\\ \bar{x}^3\end{pmatrix} = \begin{pmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x^0\\ x^1\\ x^2\\ x^3\end{pmatrix}$$

$$\begin{pmatrix}x^0\\ x^1\\ x^2\\ x^3\end{pmatrix} = \begin{pmatrix}\gamma & \gamma\beta & 0 & 0\\ \gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\bar{x}^0\\ \bar{x}^1\\ \bar{x}^2\\ \bar{x}^3\end{pmatrix}$$

$$J^\mu = (c\rho, J_x, J_y, J_z)$$

$$A^\mu = (\frac{\phi}{c}, A_x, A_y, A_z)$$

$$\bar{E}_x = E_x$$
$$\bar{E}_y = \gamma\left(E_y - \mathrm{v}\,B_z\right)$$
$$\bar{E}_z = \gamma\left(E_z + \mathrm{v}B_y\right)$$

$$\bar{B}_x = B_x$$
$$\bar{B}_y = \gamma\left(B_y + \frac{\mathrm{v}}{c^2}E_z\right)$$
$$\bar{B}_z = \gamma(B_z - \frac{\mathrm{v}}{c^2}E_y)$$

Contra-variant 4 vector transforms as

$$\begin{pmatrix}\bar{S}^0\\ \bar{S}^1\\ \bar{S}^2\\ \bar{S}^3\end{pmatrix} = \begin{pmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}S^0\\ S^1\\ S^2\\ S^3\end{pmatrix}$$

Co-variant 4 vector transforms as

$$\begin{pmatrix}\bar{S}_0\\ \bar{S}_1\\ \bar{S}_2\\ \bar{S}_3\end{pmatrix} = \begin{pmatrix}\gamma & \gamma\beta & 0 & 0\\ \gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}$$

$$\begin{pmatrix}S^0\\ S^1\\ S^2\\ S^3\end{pmatrix} = \begin{pmatrix}-S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = (-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

$$\partial_\mu F^{\mu\nu} = -\mu_o J^\nu$$

$$\partial_\mu G^{\mu\nu} = 0$$

$$X^\mu(t) = \begin{pmatrix} c\,t \\ \mathbf{X}(t) \end{pmatrix}$$

$$\mathbf{u}(t) = \frac{d\,\mathbf{X}(t)}{dt}$$

$$d\tau = d\,t\sqrt{\left(1 - \frac{\mathbf{u}^2(t)}{c^2}\right)}$$

$$\eta^\mu = \frac{d}{d\tau} X^\mu(\tau) = \begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix}$$

$$m\frac{d}{d\tau}\eta^\mu = qF^{\mu\sigma}\eta_\sigma$$

$$m\frac{d}{d\tau}\begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix} = m\gamma_u \frac{d}{dt}\begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix}$$

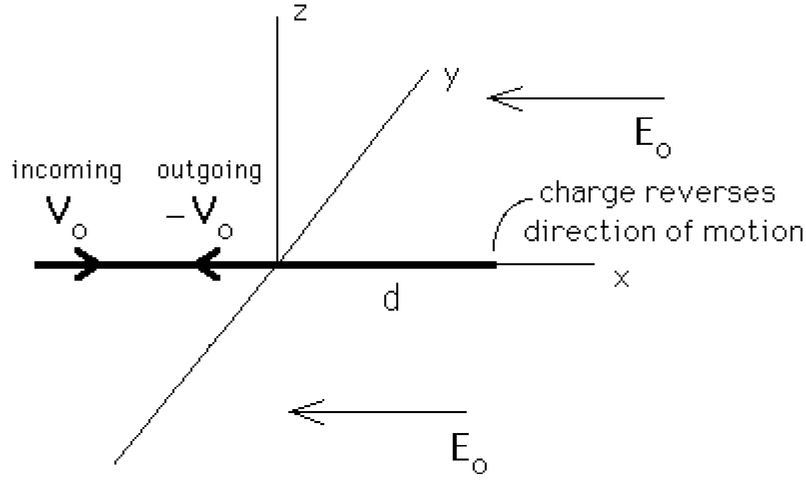
$$= q\begin{pmatrix} \gamma_u \dfrac{\mathbf{E}\cdot\mathbf{u}}{c} \\ \gamma_u \left(\mathbf{E} + \mathbf{u}\,\mathbf{x}\,\mathbf{B}\right) \end{pmatrix}$$

$$Energy\ of\ \ particle = m\gamma_u\, c^2 = \frac{m\,c^2}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Problem 1: Radiation by a non-relativistic charge

We have a static electric field $\mathbf{E}_o$ which is given by

$$\mathbf{E}_o = \begin{cases} -E_o\hat{\mathbf{x}} & x > 0 \\ 0 & x < 0 \end{cases}$$



A particle of mass *m* and charge *q* is initially located on the negative x-axis, with velocity $\mathbf{V}_o = \hat{\mathbf{x}}\, V_o$, where $V_o << c$.

At $t = 0$, the particle crosses into the region $x > 0$, feels the repulsive force due to the electric field there, and begins to decelerate at a constant rate. It penetrates into a distance *d* along the positive x axis before its velocity reverses direction, and then eventually exits the region $x > 0$, returning back down the x-axis at speed $-\mathbf{V}_o = -\hat{\mathbf{x}}\, V_o$. The time *T* that the particle spends in the region $x > 0$ is given by $T = \dfrac{2\, m V_o}{q\, E_o}$. The depth of penetration *d* into the region $x > 0$ is given by $d = \dfrac{1}{2}\dfrac{mV_o^{\ 2}}{q\, E_o}$, assuming that there is no energy loss due to radiation.

(a) An observer is located along the positive y-axis at a distance $D >> d$. This observer will see a radiation electric field which has only one non-zero component, E_x or E_y or E_z. Which is it?

(b) Sketch in the space below the non-zero radiation electric field component as seen by this observer, as a function of the observer's time *t*. Label your time axis and electric field component axes appropriately. When you label your electric field component axis, put your label(s) in terms of E_o, *q*, *m*, *D*, and fundamental constants.

(c) What is the total energy radiated into electric dipole radiation in this process? Give your answer in terms of *q*, *m*, E_o, V_o, and fundamental constants.

(d) Let the classical radius of the electron be $r_e = \frac{1}{4\pi\varepsilon_o}\frac{q^2}{m\,c^2}$. What is the ratio of the total radiated energy from part (c) to the initial kinetic energy of the particle? Write this ratio in terms of r_e, *d*, V_o and *c* alone. [Hint: your answer should have no dimensions--that is, it should only involve the ratios of lengths, speeds, etc.].

Problem 2: Relativistic dynamics of a magnetic monopole

You hear on the evening news that a magnetic dipole has finally been discovered, and you quickly go to Griffith to look up the manifestly covariant equation for relativistic dynamics for magnetic dipoles, but Griffith does not have it. You sit down and try to figure it out yourself.

If the "magnetic monopole point charge" is q_m, and its mass is m_m, you intuit that ***if you have only a magnetic field***, the non-relativistic equation of motion will be $m_m \mathbf{a} = q_m \mathbf{B}$. You want to go from this to the fully relativistic equation, including the effect of electric fields.

(a) What does your guess for the 4 vector equation for the relativistic dynamics of the magnetic monopole looks like in manifestly covariant form?   Let the 4 vector position of the magnetic monopole be $X^\mu(t) = \begin{pmatrix} ct \\ \mathbf{X}(t) \end{pmatrix}$, its ordinary velocity be $\mathbf{u}(t) = \frac{d\,\mathbf{X}(t)}{dt}$, and its 4 vector velocity be $\eta^\mu = \frac{d}{d\tau} X^\mu(\tau)$. Choose the constants in your manifestly covariant equation so that you recover $m_m \mathbf{a} = q_m \mathbf{B}$ in the non-relativistic case with no electric field.

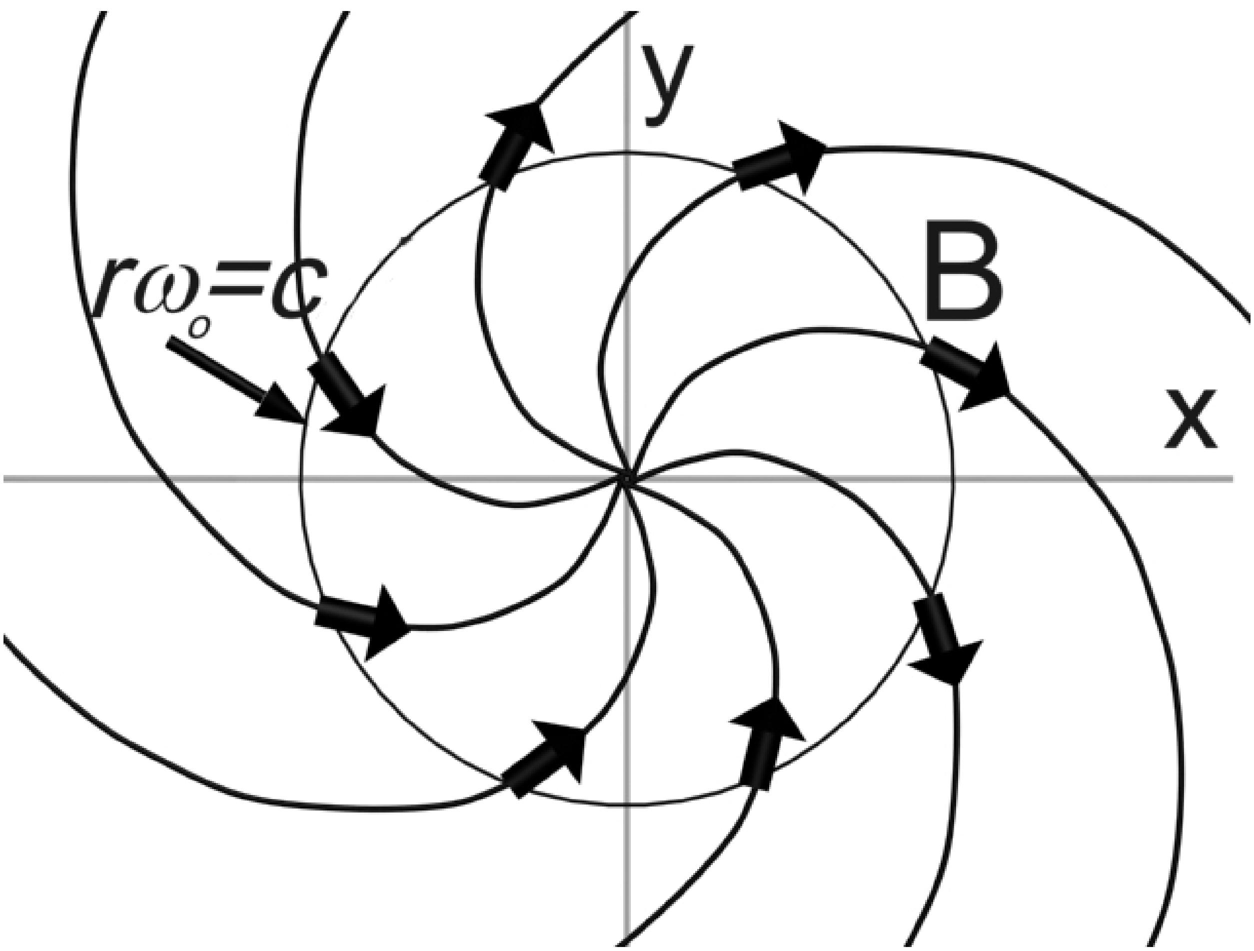
(b) What are the equations from (a) in ordinary coordinate form, that is in terms of the ordinary position and velocity of the monopole, and in terms of ordinary coordinate time derivatives. Give both the time and space components of your four vector equation from (a).

Problem 3: Angular momentum loss for a rotating magnetic dipole

In Problem 4.4 of Problem Set 4, you showed that the magnetic field in the *x*-*y* plane of a magnetic dipole lying in the *x*-*y* plane and rotating about the *z*-axis was given by the following expression far from the dipole:

$$\mathbf{B}(\mathbf{r},t)\Big|_{\theta=\pi/2} = \frac{\mu_o \omega_o m_o \sin\left[\phi - \omega_o\left(t - r/c\right)\right]}{4\pi r^2 c^2}\left[c\hat{\mathbf{r}} - r\omega_o\hat{\boldsymbol{\phi}}\right]$$

These field lines at once instant of time are plotted in the figure below. We are looking down on the x-y plane from above. The circle is drawn in the *xy* plane at a value of *r* such that $r\omega_o = c$. Over one-half of the circle the magnetic field lines point outward, and over the other half of the circle the magnetic field lines point inward, as indicated on the plot by the thick arrows.



(a) In the Figure above, at the base of every thick arrow, indicate the direction of $\overleftrightarrow{\mathbf{T}}_{\mathbf{B}} \cdot \hat{\mathbf{r}}$.

(b) The vector angular momentum flux density in the $\hat{\mathbf{r}}$ direction at the base of any of the thick arrows is given by $-\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\cdot\hat{\mathbf{r}}\right)$. What is the direction of this angular momentum flux density on the half of the circle where the magnetic field **B** points inward? What is its direction on the half of the circle where **B** points outward?

(c) Now consider the case that r has any value, not limited to $r\omega_o = c$. Give an expression for $-\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\cdot\hat{\mathbf{r}}\right)$, using the equation for $\mathbf{B}(\mathbf{r},t)\big|_{\theta=\pi/2}$ on the previous page. How does your expression for this quantity fall with r? Is this a fall-off which will result in a net angular momentum flux to infinity? Why?

NAME:______________________________________

Problem 4: Relativistic particle motion

(a) A spaceship with speed $V < c$ moves directly toward an observer in the laboratory frame, firing laser blasts with a time separation $\Delta t'$ at the observer, where $\Delta t'$ is the time separation of the blasts as measured by our infinite grid of observers at rest in the laboratory frame. What is the time separation Δt between these laser blasts when they arrive at the observer, in terms of V, c and $\Delta t'$? Justify your answer.

(b) The four acceleration vector is defined by $\Xi^\mu = \frac{d}{d\tau}\eta^\mu$ and is given in terms of ordinary acceleration and velocity by $\Xi^\mu = \gamma_u{}^2 \begin{pmatrix} \gamma_u{}^2 \left(\frac{\mathbf{u}\cdot\mathbf{a}}{c} \right) \\ \mathbf{a}(t) + \gamma_u{}^2 \mathbf{u} \left(\frac{\mathbf{u}\cdot\mathbf{a}}{c^2} \right) \end{pmatrix}$. From this expression for Ξ^μ, how

NAME:___________________________________

would you compute the acceleration of the particle in its instantaneous rest frame given quantities measured in the lab frame? This is not a question about Lorentz transformations, it is a question about Lorentz invariants. You do not have to carry out the computation, just indicate how you would do it.