Physics 8.07 **Fall 2010**

DETACH THESE SHEETS & USE AT YOUR CONVENIENCE

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_o}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_o \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\int_{\substack{\text{closed}\\\text{surface}}} \mathbf{E} \cdot \hat{\mathbf{n}}\, da = \frac{q_{\text{enclosed}}}{\varepsilon_o}$$

$$\int_{\substack{\text{bounding}\\\text{contour}}} \mathbf{E} \cdot d\,\mathbf{l} = -\frac{d}{dt} \int_{\substack{\text{open}\\\text{surface}}} \mathbf{B} \cdot \hat{\mathbf{n}}\, da$$

$$\int_{\substack{\text{bounding}\\\text{contour}}} \mathbf{B} \cdot d\,\mathbf{l} = \mu_o \varepsilon_o \left[I_{\text{through}} + \frac{d}{dt} \int_{\substack{\text{open}\\\text{surface}}} \mathbf{E} \cdot \hat{\mathbf{n}}\, da \right]$$

$$\int_{\substack{\text{closed}\\\text{surface}}} \mathbf{B} \cdot \hat{\mathbf{n}}\, da = 0$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \mathbf{A} = -\mu_o \mathbf{J}$$

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \phi = -\frac{\rho}{\varepsilon_o}$$

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$$

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_o} \int_{all\,time} dt' \int_{all\,space} \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \times \delta(t - t' - |\mathbf{r} - \mathbf{r}'|/c) d^3x'$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_o}{4\pi} \int_{all\,time} dt' \int_{all\,space} \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \times \delta(t - t' - |\mathbf{r} - \mathbf{r}'|/c) d^3x'$$

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\,\varepsilon_o} \int \rho(\mathbf{r}', t'_{ret}) \frac{d^3x'}{|\mathbf{r} - \mathbf{r}'|}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_o}{4\pi} \int \mathbf{J}(\mathbf{r}', t'_{ret}) \frac{d^3x'}{|\mathbf{r} - \mathbf{r}'|}$$

$$t'_{ret} = t - |\mathbf{r} - \mathbf{r}'|/c$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{r^3} \frac{[3\hat{\mathbf{n}}(\mathbf{p} \cdot \hat{\mathbf{n}}) - \mathbf{p}]}{4\pi\,\varepsilon_o} + \frac{1}{c\,r^2} \frac{[3\hat{\mathbf{n}}(\dot{\mathbf{p}} \cdot \hat{\mathbf{n}}) - \dot{\mathbf{p}}]}{4\pi\,\varepsilon_o} + \frac{1}{rc^2} \frac{(\ddot{\mathbf{p}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}}{4\pi\,\varepsilon_o}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_o}{4\pi} \left[\frac{\dot{\mathbf{p}}}{r^2} + \frac{\ddot{\mathbf{p}}}{c\,r} \right] \times \hat{\mathbf{n}}$$

all evaluated at $t' = t - r/c$

$$\frac{\partial}{\partial t} \left[\frac{1}{2} \varepsilon_o E^2 + \frac{B^2}{2\mu_o} \right] + \nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_o} \right) = -\mathbf{E} \cdot \mathbf{J}$$

Energy density of field = $\frac{1}{2} \varepsilon_o E^2 + \frac{B^2}{2\mu_o}$

Energy flux density = $\frac{\mathbf{E} \times \mathbf{B}}{\mu_o}$

$$\frac{\partial}{\partial t} [\varepsilon_o \mathbf{E} \times \mathbf{B}] + \nabla \cdot \left(-\overleftrightarrow{\mathbf{T}} \right) = -[\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}]$$

Momentum density = $\varepsilon_o \mathbf{E} \times \mathbf{B}$

Momentum flux density = $-\overleftrightarrow{\mathbf{T}}$

$$\frac{\partial}{\partial t} [\varepsilon_o \mathbf{r} \times (\mathbf{E} \times \mathbf{B})] + \nabla \cdot \left(-\mathbf{r} \times \overleftrightarrow{\mathbf{T}} \right) = -\mathbf{r} \times [\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}]$$

Angular momentum density = $\varepsilon_o \mathbf{r} \times (\mathbf{E} \times \mathbf{B})$

Angular momentum flux density = $-\mathbf{r} \times \overleftrightarrow{\mathbf{T}}$

$$\overleftrightarrow{\mathbf{T}} = \varepsilon_o \left[\mathbf{EE} - \frac{1}{2} \overleftrightarrow{\mathbf{I}} E^2 \right] + \frac{1}{\mu_o} \left[\mathbf{BB} - \frac{1}{2} \overleftrightarrow{\mathbf{I}} B^2 \right]$$

$\overleftrightarrow{\mathbf{T}}_{\mathbf{E}} \cdot \hat{\mathbf{n}}$ lies in the plane defined by $\mathbf{E}$ and $\hat{\mathbf{n}}$.

The magnitude of $\overleftrightarrow{\mathbf{T}}_{\mathbf{E}} \cdot \hat{\mathbf{n}}$ is always $\frac{1}{2}\varepsilon_o E^2$

If you go an angle θ to get to $\mathbf{E}$ from $\hat{\mathbf{n}}$, go an angle 2θ in the same sense to get to $\overleftrightarrow{\mathbf{T}}_{\mathbf{E}} \cdot \hat{\mathbf{n}}$

All of the above apply to $\overleftrightarrow{\mathbf{T}}_{\mathbf{B}} \cdot \hat{\mathbf{n}}$ except for the fact that $\left| \overleftrightarrow{\mathbf{T}}_{\mathbf{B}} \cdot \hat{\mathbf{n}} \right|$ is always $B^2 / 2\mu_o$

$$\mathbf{p}(t') = \int \mathbf{r}' \rho(\mathbf{r}', t') d^3x'$$

$$\frac{dW}{d\Omega \ dt} = \frac{c}{\mu_o} (\mathbf{E} \times \mathbf{B}) \cdot (r^2 \ \hat{\mathbf{r}})$$

$$\frac{dW_{el\ dip}}{d\Omega \ dt} = \frac{c \ r^2}{\mu_o} \left[\frac{\mu_o}{4\pi} \frac{\ddot{\mathbf{p}} \, \mathbf{x} \, \hat{\mathbf{n}}}{c \ r} \right]^2 = \frac{\mu_o \ \ddot{p}^{\,2}}{(4\pi)^2 c} \sin^2 \theta$$

$$\frac{dW_{el\ dip}}{dt} = \frac{1}{4\pi \varepsilon_o} \frac{2}{3} \frac{|\ddot{\mathbf{p}}|^2}{c^3}$$

$$\frac{dW_{el\ dip}}{dt} = \frac{1}{4\pi \varepsilon_o} \frac{2}{3} \frac{q^2 a^2}{c^3}$$

$$\mathbf{m}(t') = \frac{1}{2} \int \mathbf{r}' \times \mathbf{J}(\mathbf{r}', t') d^3x'$$

$$\frac{dW_{mag\ dip}}{dt} = \frac{\mu_o}{4\pi} \frac{2|\ddot{\mathbf{m}}|^2}{3c^3}$$

$$\rho(\mathbf{r}, t) = q\delta^3 (\mathbf{r} - \mathbf{X}(t))$$

$$\mathbf{J}(\mathbf{r}, t) = q\mathbf{u}(t)\delta^3 (\mathbf{r} - \mathbf{X}(t))$$

$$\mathbf{u}(t) = \frac{d}{dt} \mathbf{X}(t)$$

$$\mathbf{p}(t') = \int \mathbf{r}' \rho(\mathbf{r}', t') d^3x' = q\mathbf{X}(t')$$

$$\mathbf{m}(t') = \frac{1}{2} \int \mathbf{r}' \times \mathbf{J}(\mathbf{r}', t') d^3x' = \frac{q}{2} \mathbf{X}(t') \times \mathbf{u}(t')$$

8.07 Midterm exam Oct 22, 2010

Name: ____

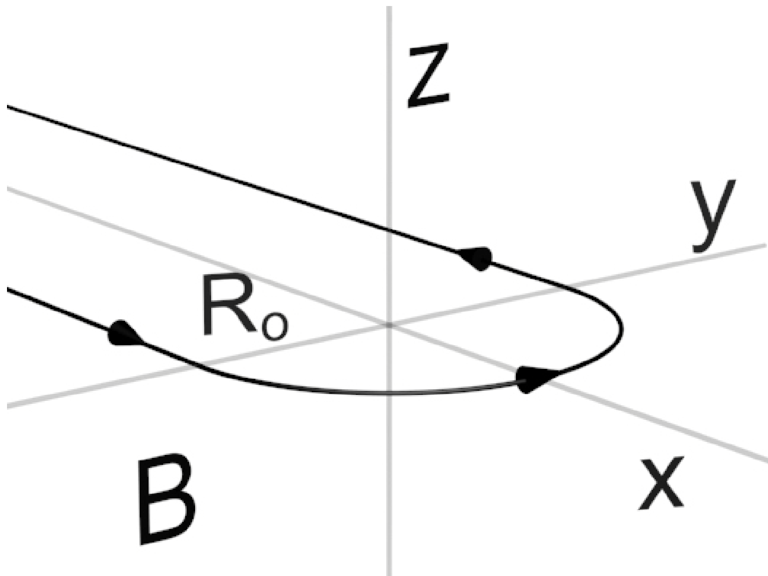
Problem	Grade	Grader
1		
2		
Total		

Problem 1: Radiation from a non-relativistic charge in a constant magnetic field

We have a magnetic field **B** which is independent of time but varies in space as indicated. A particle of mass *m* and charge *q* is initially in the magnetic-field-free region $x < 0$ with velocity $\mathbf{V}_o = \hat{\mathbf{x}}\, V_o$, where $V_o << c$.

$$\mathbf{B} = \begin{cases} 0 & x < 0 \\ -B_o \hat{\mathbf{z}} & x > 0 \end{cases}$$

At $t = 0$, the particle crosses into the region $x > 0$, feels the $q\,\mathbf{v}\times\mathbf{B}$ force due to the magnetic field there, moves in a semi-circle of radius R_0, and then exits the region $x > 0$ (see sketch). Assuming there is no energy loss due to radiation, the particle's position along the semi-circle as a function of time is given by



$$\mathbf{X}(t) = R_0\left[\hat{\mathbf{x}} \sin\omega_0 t - \hat{\mathbf{y}}\cos\omega_0 t\right]$$
$$0 < t < \pi/\omega_0$$

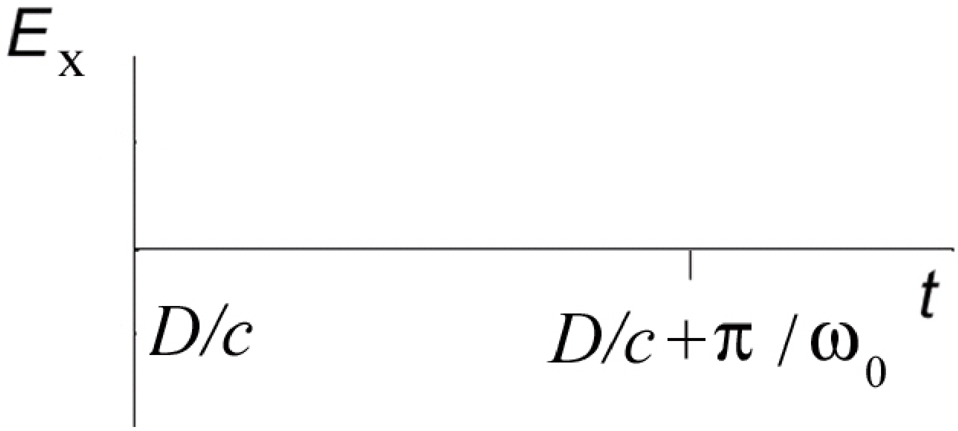
where $R_0 = V_0/\omega_0$

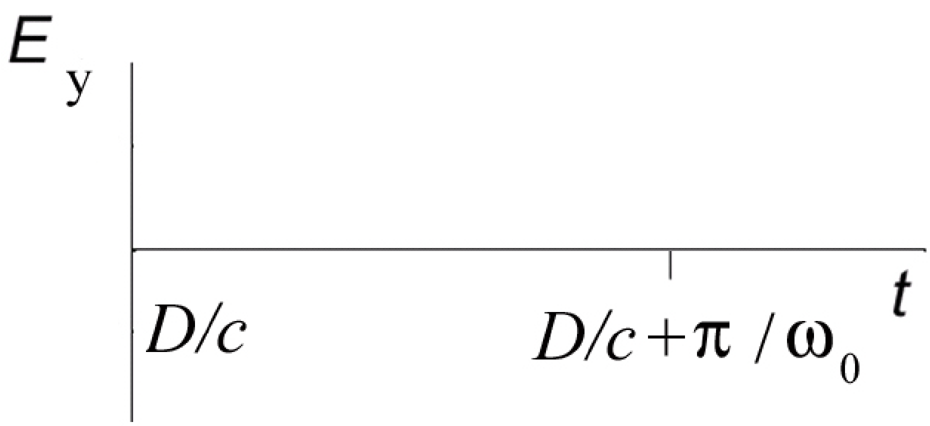
(a) Using the equation of motion of the particle when it is in the magnetic field, show that

$$\omega_0 = qB_0/m$$

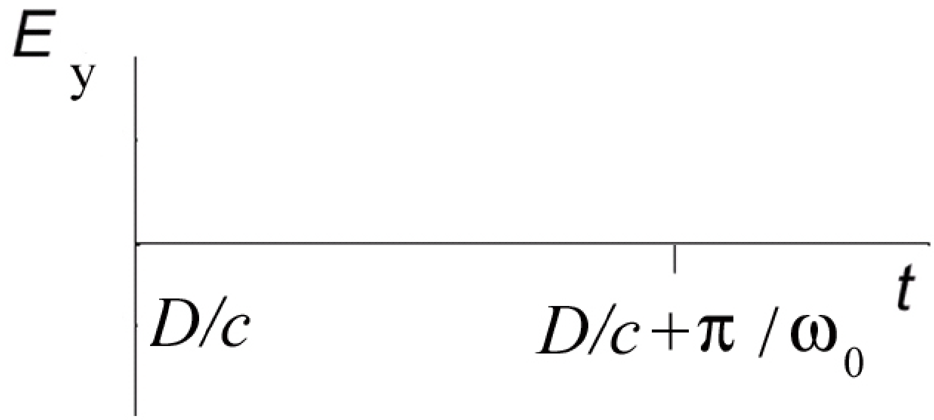
Note: if you cannot prove this equation, the rest of the problem is doable assuming it is true.

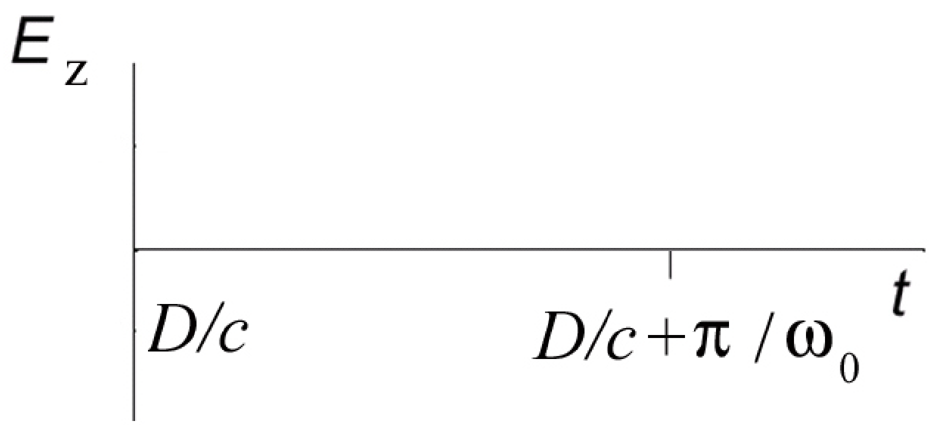
(b) An observer is located along the positive *z*-axis at a distance $D >> R_0$. This observer will see a burst of electric dipole radiation lasting a time π/ω_0. On the graphs sketch the shape of the *x*-component and the *y*-component of the electric dipole radiation field as a function of time at that observer's position. ***You do not have to indicate the absolute magnitude of the electric field components, but you do have to get the shapes correctly.***


E_x
t
D/c
$D/c + \pi/\omega_0$


E_y
t
D/c
$D/c + \pi/\omega_0$

(c) Another observer is located along the positive x-axis at a distance $D >> R_0$. This observer will also see a burst of electric dipole radiation lasting a time π / ω_0. On the graphs sketch the shape of the z-component and the y-component of the electric dipole radiation field as a function of time at that observer's position. ***You do not have to indicate the absolute magnitude of the electric field, but you do have to get the shapes correctly.***

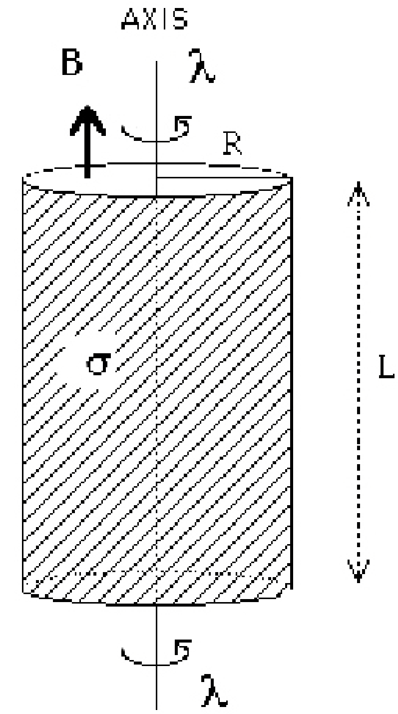




(d) What is the total energy radiated into electric dipole radiation in this process? Give your answer in terms of q, m, B_o, V_o, and fundamental constants.

(e) Let the classical radius of the electron be $r_e = \dfrac{q^2}{4\pi\,\varepsilon_o m\,c^2}$. What is the ratio of the total radiated energy from part (d) to the initial kinetic energy of the particle? Write this ratio in terms of r_e, R_0, V_o and c alone. [Hint: your answer should have no dimensions--that is, it should only involve the ratios of lengths, speeds, etc.].

Problem 2: Rotating Cylindrical Shell



A long cylindrical shell has length L and radius R, with $L >> R$. The cylindrical shell is suspended so that it can rotate freely about its axis with no friction. The cylindrical shell carries a charge per unit area σ which is glued onto its surface, where $\sigma > 0$. Along the axis of the cylinder is a line charge with charge per unit length $\lambda = -2\pi R\sigma < 0$. Thus the static electric field is given by

$$\mathbf{E}_{static}(\mathbf{r},t) = \begin{cases} -\dfrac{R}{r}\dfrac{\sigma}{\varepsilon_o}\hat{\mathbf{r}} & r < R \\ 0 & r > R \end{cases}$$

Now, we begin spinning the cylindrical shell at an angular velocity $\omega(t)$ with $\omega R << c$. The motion of the charge glued onto the surface of the spinning sphere results in a surface current

$$\boldsymbol{\kappa}(t) = \sigma\,\omega(t)\,R\,\hat{\boldsymbol{\phi}}$$

We assume that we can use the quasi-static approximation to get a good approximation to the time dependent solution for **B** (good for variations in $\kappa(t)$ with time scales $T \approx \dfrac{\kappa}{d\kappa/dt} \gg \dfrac{R}{c}$)

(a) ***Show*** using the appropriate Maxwell equation that our quasi-static solution for **B** (neglecting fringing fields) is given by the equation below. State the Maxwell equation you use and clearly show the steps in your derivation. Do not make this too complicated, this is an 8.02 question.

$$\mathbf{B}_{\text{quasi-static}}(r) = \begin{cases} \mu_o\kappa(t)\,\hat{\mathbf{z}} = \mu_o\sigma\omega(t)R\,\hat{\mathbf{z}} & r < R \\ 0 & r > R \end{cases}$$

(b) Given this quasi-static solution for **B**, ***show*** using the appropriate Maxwell equations that the electric field everywhere in space (neglecting fringing fields) is now given by

$$\mathbf{E}(\mathbf{r},t)=\begin{cases}-\dfrac{R}{r}\dfrac{\sigma}{\varepsilon_o}\hat{\mathbf{r}}-\dfrac{r}{2}\mu_o R\sigma\dfrac{d\omega}{dt}\hat{\boldsymbol{\phi}} & r<R\\[2ex] -\dfrac{R^3}{2r}\mu_o\sigma\dfrac{d\omega}{dt}\hat{\boldsymbol{\phi}} & r>R\end{cases}$$

(you need only derive the form for E_ϕ, we have already given you E_r above). State the Maxwell equation you use and clearly show the steps in your derivation. Do not make this too complicated, this is an 8.02 question.

(c) What is the total magnetic energy in terms of the parameters given?

(d) Using the solutions above, show that the total rate at which electromagnetic energy is being created as the cylinder is being spun up, $\int_{\text{all space}} -\mathbf{J}\cdot\mathbf{E}\, d^3x$, is equal to the rate at which the total magnetic energy is increasing.

(e) Using the solutions above, calculate the flux of electromagnetic energy $\int_{surface} \left[\frac{\mathbf{E} \times \mathbf{B}}{\mu_o} \right] \cdot \hat{\mathbf{r}}\, da$ through a cylindrical surface of radius *r* and height *L* for *r* a little greater than *R* and also for *r* a little smaller than *R*. Do your results agree with what you expect from (c) and (d)? Neglect fringing fields.

(f) Using the solutions above, calculate the total electromagnetic angular momentum $\int \mathbf{r}\times\left[\varepsilon_o \mathbf{E}\times\mathbf{B}\right]d^3x$ in terms of the parameters given? Neglect fringing fields.

(g) Using the solutions above, show that the total rate at which electromagnetic angular momentum is being created as the cylinder is being spun up, $\int_{\text{all space}} -\mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right]d^3x$, is equal to the rate at which the total electromagnetic angular momentum is increasing.

(h) Using the solutions above, calculate the flux of electromagnetic angular momentum, $\int_{surface} \left[-\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{r}}\right)\right]da$ through a cylinder of radius r and height L for r a little greater than R and for r a little smaller than R. Do your results agree with what you expect from (f) and (g)? As in all stress tensor calculations, figure out what components you are going to need before you calculate anything, and then just calculate those. Neglect fringing fields.