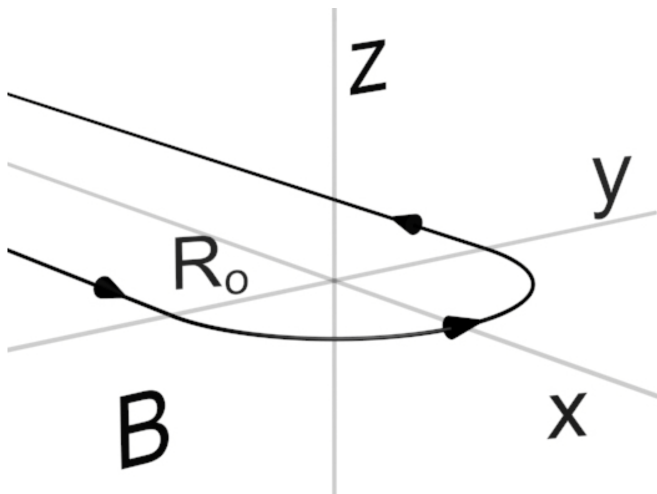
Problem 1: Radiation from a non-relativistic charge in a constant magnetic field

$$\mathbf{B} = \begin{cases} 0 & x<0 \\ -B_o\hat{\mathbf{z}} & x>0 \end{cases}$$

We have a magnetic field **B** which is independent of time but varies in space as indicated. A particle of mass *m* and charge *q* is initially in the magnetic-field-free region $x < 0$ with velocity $\mathbf{V}_o = \hat{\mathbf{x}}\, V_o$, where $V_O << c$. At $t = 0$, the particle crosses into the region $x > 0$, feels the $q\,\mathbf{v}\times\mathbf{B}$ force due to the magnetic field there, moves in a semi-circle of radius R_0, and then exits the region $x > 0$ (see sketch). Assuming there is no energy loss due to radiation, the particle's position along the semi-circle as a function of time is given by

$$\mathbf{X}(t) = R_0\left[\hat{\mathbf{x}}\sin\omega_0 t - \hat{\mathbf{y}}\cos\omega_0 t\right]$$
$$0 < t < \pi/\omega_0$$

where $R_0 = V_0/\omega_0$



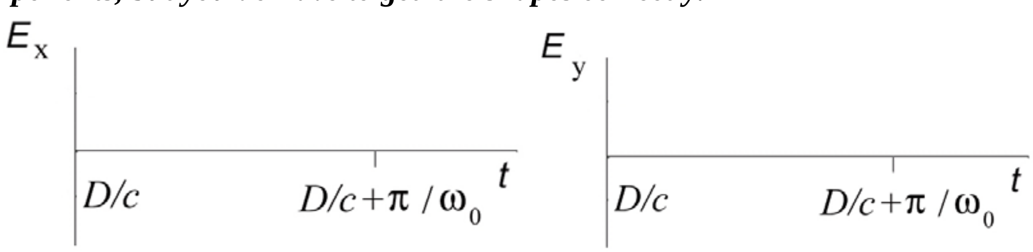
(a) Using the equation of motion of the particle when it is in the magnetic field, show that

$$\omega_0 = qB_0/m$$

Note: if you cannot prove this equation, the rest of the problem is doable assuming it is true.

Something moving in a semi-circle at constant speed accelerates inwards toward the center of the circle with acceleration $V_0^2/R_0 = \omega_0^2 R_0$. The force due to the magnetic field is $qV_0B_0 = q\omega_0 R_0 B_0$. Equating the force with *ma* using Newton yields $m\,\omega_0^2 R_0 = q\omega_0 R_0 B_0$, or $\omega_0 = qB_0/m$.

(b) An observer is located along the positive *z*-axis at a distance $D >> R_0$. This observer will see a burst of electric dipole radiation lasting a time π/ω_0. On the graphs sketch the shape of the *x*-component and the *y*-component of the electric dipole radiation field as a function of time at that observer's position. ***You do not have to indicate the absolute magnitude of the electric field components, but you do have to get the shapes correctly.***



$$\mathbf{E}_{radiation}(\mathbf{r},t) = \frac{1}{rc^2}\frac{(\ddot{\mathbf{p}}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{4\pi\,\varepsilon_o} \qquad \mathbf{p}(t) = q\mathbf{X}(t) = qR_0\left[\hat{\mathbf{x}}\sin\omega_0 t - \hat{\mathbf{y}}\cos\omega_0 t\right]$$

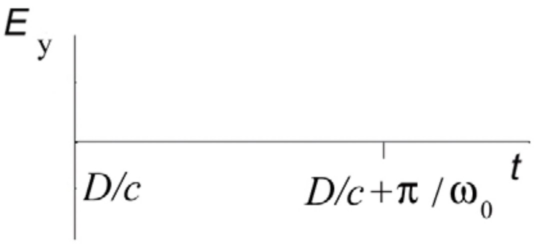
$$\ddot{\mathbf{p}}(t) = -q\omega_0^2 R_0\left[\hat{\mathbf{x}}\sin\omega_0 t - \hat{\mathbf{y}}\cos\omega_0 t\right] \qquad \hat{\mathbf{n}} = \hat{\mathbf{x}}$$

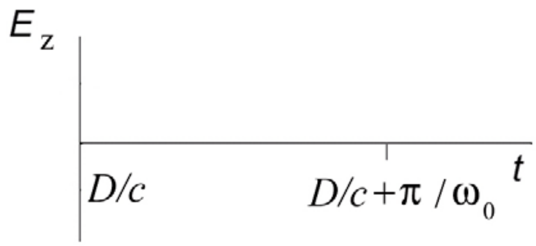
$$(\ddot{\mathbf{p}}\times\hat{\mathbf{n}}) = -q\omega_0^2 R_0\left[\hat{\mathbf{x}}\sin\omega_0 t - \hat{\mathbf{y}}\cos\omega_0 t\right]\times\hat{\mathbf{z}} = -q\omega_0^2 R_0\sin\omega_0 t\,\hat{\mathbf{x}}\times\hat{\mathbf{z}} + q\omega_0^2 R_0\cos\omega_0 t\,\hat{\mathbf{y}}\times\hat{\mathbf{z}}$$
$$= q\omega_0^2 R_0\sin\omega_0 t\,\hat{\mathbf{y}} + q\omega_0^2 R_0\cos\omega_0 t\,\hat{\mathbf{x}}$$

$$(\ddot{\mathbf{p}}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}} = q\omega_0^2 R_0\left(\sin\omega_0 t\,\hat{\mathbf{y}} + \cos\omega_0 t\,\hat{\mathbf{x}}\right)\times\hat{\mathbf{z}} = q\omega_0^2 R_0\left(\sin\omega_0 t\,\hat{\mathbf{x}} - \cos\omega_0 t\,\hat{\mathbf{y}}\right)$$

So the x-component is the first half of a sin wave, and the y component if the first half of a negative cosine wave.

(c) Another observer is located along the positive x-axis at a distance $D >> R_0$. This observer will also see a burst of electric dipole radiation lasting a time π/ω_0. On the graphs sketch the shape of the z-component and the y-component of the electric dipole radiation field as a function of time at that observer's position. ***You do not have to indicate the absolute magnitude of the electric field, but you do have to get the shapes correctly.***





$$(\ddot{\mathbf{p}}\times\hat{\mathbf{n}}) = -q\omega_0^2 R_0\left[\hat{\mathbf{x}}\sin\omega_0 t - \hat{\mathbf{y}}\cos\omega_0 t\right]\times\hat{\mathbf{x}} = +q\omega_0^2 R_0\cos\omega_0 t\,\hat{\mathbf{y}}\times\hat{\mathbf{x}} = -q\omega_0^2 R_0\cos\omega_0 t\,\hat{\mathbf{z}}$$

$$(\ddot{\mathbf{p}}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}} = -q\omega_0^2 R_0\cos\omega_0 t\,\hat{\mathbf{z}}\times\hat{\mathbf{x}} = -q\omega_0^2 R_0\cos\omega_0 t\,\hat{\mathbf{y}}$$

So the y component is the first half of a negative cos wave and the x component is zero.

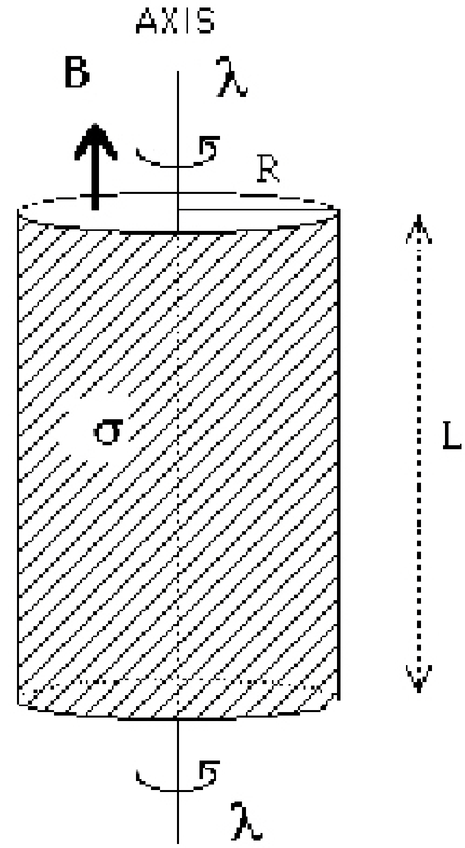
(d) What is the total energy radiated into electric dipole radiation in this process? Give your answer in terms of q, m, B_o, V_o, and fundamental constants.

$$\frac{dW_{el\,dip}}{dt} = \frac{1}{4\pi\,\varepsilon_o}\frac{2}{3}\frac{\left|\ddot{\mathbf{p}}\right|^2}{c^3} = \frac{1}{4\pi\,\varepsilon_o}\frac{2}{3c^3}\left(q\omega_0^2 R_0\right)^2\left[\left(\sin\omega_0 t\right)^2 + \left(\cos\omega_0 t\right)^2\right] = \frac{q^2\omega_0^4 R_0^2}{6\pi\,\varepsilon_o c^3}$$

$$W_{el\,dip} = \frac{q^2\omega_0^4 R_0^2}{6\pi\,\varepsilon_o c^3}\frac{\pi}{\omega_0} = \frac{q^2\omega_0^3 R_0^2}{6\,\varepsilon_o c^3} = \frac{q^2\omega_0 V_0^2}{6\,\varepsilon_o c^3} = \frac{q^3 B_0 V_0^2}{6\,\varepsilon_o c^3 m}$$

(e) Let the classical radius of the electron be $r_e = \dfrac{q^2}{4\pi\,\varepsilon_o m\,c^2}$. What is the ratio of the total radiated energy from part (d) to the initial kinetic energy of the particle? Write this ratio in terms of r_e, R_0, V_o and c alone. [Hint: your answer should have no dimensions--that is, it should only involve the ratios of lengths, speeds, etc.].

$$\frac{W_{el\,dip}}{\frac{1}{2}mV_0^2} = \frac{q^3 B_0}{3\varepsilon_o c^3 m^2} = \frac{q^2}{3\varepsilon_o c^3 m}\omega_0 = \frac{4\pi}{3}\frac{V_0}{c}\frac{r_e}{R_0}$$



Problem 2: Rotating Cylinder

A long cylinder has length L and radius R, with $L >> R$. The cylinder is suspended so that it can rotate freely about its axis with no friction. The cylinder carries a charge per unit area σ which is glued onto its surface, where $\sigma > 0$. Along the axis of the cylinder is a line charge with charge per unit length $\lambda = -2\pi R\sigma < 0$. Thus the static electric field is given by

$$\mathbf{E}_{static}(\mathbf{r},t) = \begin{cases} -\dfrac{R}{r}\dfrac{\sigma}{\varepsilon_o}\hat{\mathbf{r}} & r < R \\ 0 & r > R \end{cases}$$

Now, we begin spinning the cylinder at a angular velocity $\omega(t)$ with $\omega R << c$. The motion of the charge glued onto the surface of the spinning sphere results in a surface current

$$\boldsymbol{\kappa}(t) = \sigma\,\omega(t)\,R\,\hat{\boldsymbol{\phi}}$$

We assume that we can use the quasi-static approximation to get a good approximation to the time dependent solution for **B** (good for variations in $\kappa(t)$ with time scales $T \approx \dfrac{\kappa}{d\kappa/dt} \gg \dfrac{R}{c}$)

(a) ***Show*** using the appropriate Maxwell equation that our quasi-static solution for **B** (neglecting fringing fields) is given by the equation below. State the Maxwell equation you use and clearly show the steps in your derivation. Do not make this too complicated, this is an 8.02 question.

$$\mathbf{B}_{\text{quasi-static}}(r) = \begin{cases} \mu_o\kappa(t)\,\hat{\mathbf{z}} & r < R \\ 0 & r > R \end{cases} = \begin{cases} \mu_o\sigma\omega(t)R\hat{\mathbf{z}} & r < R \\ 0 & r > R \end{cases}$$

Use the integral form of Ampere's Law in the usual 8.02 fashion. That is, take a rectangular loop with long vertical sides of length l and horizontal sides of length w, with the vertical sides parallel to the axis of the cylinder and the loop half in and half out of the cylinder. The line integral of **B** will give Bl and the current through the loop is kappa l, so that we have the equation above.

(b) Given this quasi-static solution for **B**, ***show*** using the appropriate Maxwell equations that the electric field everywhere in space (neglecting fringing fields) is now given by

$$\mathbf{E}(\mathbf{r},t) = \begin{cases} -\dfrac{R}{r}\dfrac{\sigma}{\varepsilon_o}\hat{\mathbf{r}} - \dfrac{r}{2}\mu_o R\sigma\dfrac{d\omega}{dt}\hat{\boldsymbol{\phi}} & r < R \\ -\dfrac{R^3}{2r}\mu_o\sigma\dfrac{d\omega}{dt}\hat{\boldsymbol{\phi}} & r > R \end{cases}$$

(you need only derive the form for E_ϕ, we have already given you E_r above). State the

Maxwell equation you use and clearly show the steps in your derivation. Do not make this too complicated, this is an 8.02 question.

Use the integral form of Faraday's Law in the usual 8.02 fashion. That is take a circle of radius r in the xy plane, centered at the origin. The line integral of **E** will always give $2\pi r E_\phi$ and the enclosed magnetic flux will either be $\pi r^2 B = \pi r^2 \mu_o \sigma\omega(t) R$ or $\pi R^2 B = \pi R^2 \mu_o \sigma\omega(t) R$, depending on whether r is smaller than or larger than R.

(c) What is the total magnetic energy in terms of the parameters given?

$$W_M = \int \frac{B^2}{2\mu_o} d^3x = L\pi R^2 \frac{B^2}{2\mu_o} = \frac{1}{2} L\pi R^4 \mu_o \sigma^2 \omega^2$$

(d) Using your solutions above, show that the total rate at which electromagnetic energy is being created as the cylinder is being spun up, $\int_{\text{all space}} -\mathbf{J}\cdot\mathbf{E}\, d^3x$, is equal to the rate at which the total magnetic energy is increasing.

$$\int_{\text{all space}} -\mathbf{J}\cdot\mathbf{E}\, d^3x =$$

$$\int_{surface} -\boldsymbol{\kappa}\cdot\mathbf{E}\, da = \int_{surface} -\sigma\omega R E_\phi\, da = +L 2\pi R^2 \sigma\omega\left(\frac{R^2}{2}\mu_o \sigma \frac{d\omega}{dt}\right)$$

$$\int_{\text{all space}} -\mathbf{J}\cdot\mathbf{E}\, d^3x = \frac{d}{dt}\frac{1}{2}\mu_o L\pi R^4 \sigma^2 \omega^2 = \frac{dW_M}{dt}$$

(e) Using your solutions above, calculate the flux of electromagnetic energy $\int_{surface}\left[\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right]\cdot\hat{\mathbf{r}}\, da$ through a cylindrical surface of radius r and height L for r a little greater than R and also for r a little smaller than R. Do your results agree with what you expect from (c) and (d)? Neglect fringing fields.

$\int_{surface}\left[\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right]\cdot\hat{\mathbf{r}}\, da = 0$ for r a little greater than R since there is no magnetic field there. For r a little smaller than R, we have

$$\int_{surface}\left[\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right]\cdot\hat{\mathbf{r}}\, da = 2\pi R L\left\{\left[-\frac{\sigma}{\varepsilon_o}\hat{\mathbf{r}} - \frac{R^2}{2}\mu_o\sigma\frac{d\omega}{dt}\hat{\boldsymbol{\phi}}\right]\times R\sigma\omega\,\hat{\mathbf{z}}\right\}\cdot\hat{\mathbf{r}} = -\pi R^4 L \mu_o \sigma^2 \omega \frac{d\omega}{dt} = -\frac{dW_M}{dt}$$

Where the negative sign indicates energy flow inward. This agrees with our results above.

(f) Using your solutions above, calculate the total electromagnetic angular momentum $\int \mathbf{r}\times\left[\varepsilon_o \mathbf{E}\times\mathbf{B}\right] d^3x$ in terms of the parameters given? Neglect fringing fields.

$$\mathbf{r}\times\left[\varepsilon_o\mathbf{E}\times\mathbf{B}\right] = \mathbf{r}\times\begin{cases}\left[-\frac{R}{r}\sigma\hat{\mathbf{r}} - \frac{r}{2}\varepsilon_o\mu_o\sigma R\frac{d\omega}{dt}\hat{\boldsymbol{\phi}}\right]\times\mu_o\kappa\,\hat{\mathbf{z}} & r < R \\ 0 & r > R\end{cases}$$

$$\mathbf{r}\times\left[\varepsilon_o\mathbf{E}\times\mathbf{B}\right]=\mu_o\kappa\,\mathbf{r}\times\begin{cases}\left[\dfrac{R}{r}\sigma\hat{\boldsymbol{\phi}}-\dfrac{r}{2c^2}R\sigma\dfrac{d\omega}{dt}\hat{\mathbf{r}}\right] & r<R\\ 0 & r>R\end{cases}$$

$$\mathbf{r}\times\left[\varepsilon_o\mathbf{E}\times\mathbf{B}\right]=\begin{cases}\left[R^2\mu_o\sigma^2\omega\hat{\mathbf{z}}\right] & r<R\\ 0 & r>R\end{cases}$$

$$\mathbf{L}_{EM}=\int\mathbf{r}\times\left[\varepsilon_o\mathbf{E}\times\mathbf{B}\right]d^3x=\hat{\mathbf{z}}\int_0^L dz\int_0^{2\pi}d\phi\int_0^R r\,dr\left(R^2\mu_o\sigma^2\omega\right)=\hat{\mathbf{z}}L2\pi\frac{1}{2}R^4\mu_o\sigma^2\omega=\hat{\mathbf{z}}L\pi R^4\mu_o\sigma^2\omega$$

(g) Using your solutions above, show that the total rate at which electromagnetic angular momentum is being created as the cylinder is being spun up, $\int_{\text{all space}}-\mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right]d^3x$, is equal to the rate at which the total electromagnetic angular momentum is increasing.

$$\int_{\text{all space}}-\mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right]d^3x=$$

$$\int_{surface}-\mathbf{r}\times\begin{bmatrix}\sigma\mathbf{E}\\+\boldsymbol{\kappa}\times\mathbf{B}\end{bmatrix}da=$$

$$=-\int_{surface}\mathbf{r}\times\sigma\mathbf{E}\,da \qquad \text{since } \boldsymbol{\kappa}\times\mathbf{B} \text{ is radial.}$$

$$\int_{surface}-\mathbf{r}\times\sigma\mathbf{E}\,da=-\hat{\mathbf{z}}\int_{surface}r\,\sigma E_\phi\,da=\hat{\mathbf{z}}2\pi RL\frac{R^3}{2}\mu_o\sigma^2\frac{d\omega}{dt}=\frac{d}{dt}\hat{\mathbf{z}}\pi R^4L\mu_o\sigma^2\omega=\frac{d}{dt}\mathbf{L}_{EM}$$

(h) Using your solutions above, calculate the flux of electromagnetic angular momentum, $\int_{surface}\left[-\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{r}}\right)\right]da$ through a cylinder of radius *r* and height *L* for *r* a little greater than *R* and for *r* a little smaller than *R*. Do your results agree with what you expect from (f) and (g)? As in all stress tensor calculations, figure out what components you are going to need before you calculate anything, and then just calculate those. Neglect fringing fields.

$$\int_{surface}\left[-\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{r}}\right)\right]da=-\int_{surface}\left[\mathbf{r}\times\left(T_{rr}\hat{\mathbf{r}}+T_{r\phi}\hat{\boldsymbol{\phi}}+T_{rz}\hat{\mathbf{z}}\right)\right]da=-\hat{\mathbf{z}}\int_{surface}rT_{r\phi}\,da=-\hat{\mathbf{z}}\int_{surface}r\varepsilon_oE_rE_\phi\,da$$

For *r* a little greater than *R* the flux is zero because the radial component of the electric field is zero there. For *r* a little smaller than *R*,

$$\int_{surface}\left[-\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{r}}\right)\right]da=-\hat{\mathbf{z}}\int_{surface} r\varepsilon_o\left(-\frac{\sigma}{\varepsilon_o}\right)\left(-\frac{R^2}{2}\mu_o\sigma\frac{d\omega}{dt}\right)da=-\hat{\mathbf{z}}R\varepsilon_o\frac{\sigma}{\varepsilon_o}\frac{R^2}{2}\mu_o\sigma\frac{d\omega}{dt}2\pi RL$$

$$=-\frac{d}{dt}\hat{\mathbf{z}}\pi R^4L\mu_o\sigma^2\omega=-\frac{d}{dt}\mathbf{L}_{EM}$$

The negative sign here is ok because this is the flux in the positive radial direction, the flux inward is just the opposite of that, which is what we want.