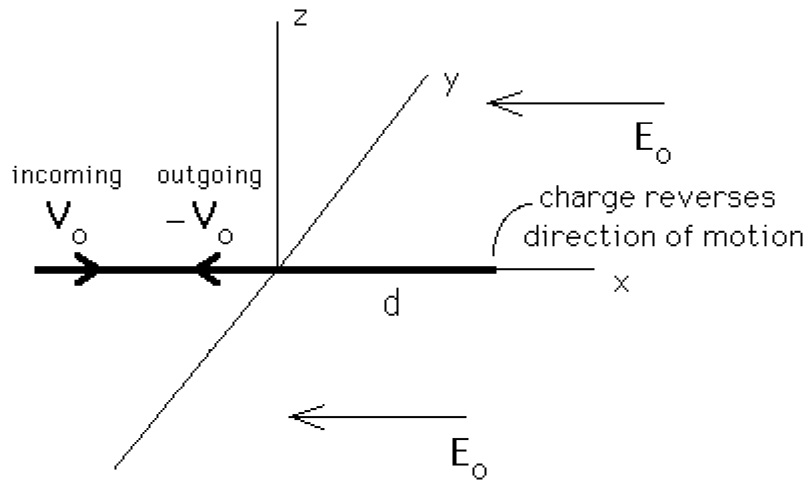
Problem 1: Radiation by a non-relativistic charge

We have a static electric field $\mathbf{E}_o$ which is given by

$$\mathbf{E}_o = \begin{cases} -E_o\hat{\mathbf{x}} & x > 0 \\ 0 & x < 0 \end{cases}$$



A particle of mass m and charge q is initially located on the negative x-axis, with velocity $\mathbf{V}_o = \hat{\mathbf{x}}\, V_o$, where $V_o << c$.

At $t = 0$, the particle crosses into the region $x > 0$, feels the repulsive force due to the electric field there, and begins to decelerate at a constant rate. It penetrates into a distance d along the positive x axis before its velocity reverses direction, and then eventually exits the region $x > 0$, returning back down the x-axis at speed $-\mathbf{V}_o = -\hat{\mathbf{x}}\, V_o$. The time T that the particle spends in the region $x > 0$ is given by $T = \dfrac{2\,m V_o}{q\,E_o}$. The depth of penetration d into the region $x > 0$ is given by $d = \dfrac{1}{2}\dfrac{mV_o^{\,2}}{q\,E_o}$, assuming that there is no energy loss due to radiation.

(a) An observer is located along the positive y-axis at a distance $D >> d$. This observer will see a radiation electric field which has only one non-zero component, E_x or E_y or E_z. Which is it?

We can do this part two different ways. The first treats our formulas for electric dipole radiation, with the second time derivative of the electric dipole moment in this case given by $\ddot{\mathbf{p}} = q\mathbf{a}$. From the formula sheet we have that during the time that the particle is accelerating that $\mathbf{E}(\mathbf{r},t) = \dfrac{1}{rc^2}\dfrac{(\ddot{\mathbf{p}}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{4\pi\,\varepsilon_o}$ with $\ddot{\mathbf{p}} = q\mathbf{a} = \dfrac{q}{m}\mathbf{E} = -\hat{\mathbf{x}}\dfrac{qE_o}{m}$. So during that time we have

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{Dc^2}\frac{(-\hat{\mathbf{x}}\dfrac{qE_o}{m}\times\hat{\mathbf{y}})\times\hat{\mathbf{y}}}{4\pi\,\varepsilon_o} = \frac{qE_o}{4\pi\,\varepsilon_o m}\frac{1}{Dc^2}(-\hat{\mathbf{z}})\times\hat{\mathbf{y}} = \frac{qE_o}{4\pi\,\varepsilon_o m}\frac{1}{Dc^2}\hat{\mathbf{x}}$$

So the radiation field only has an x component.

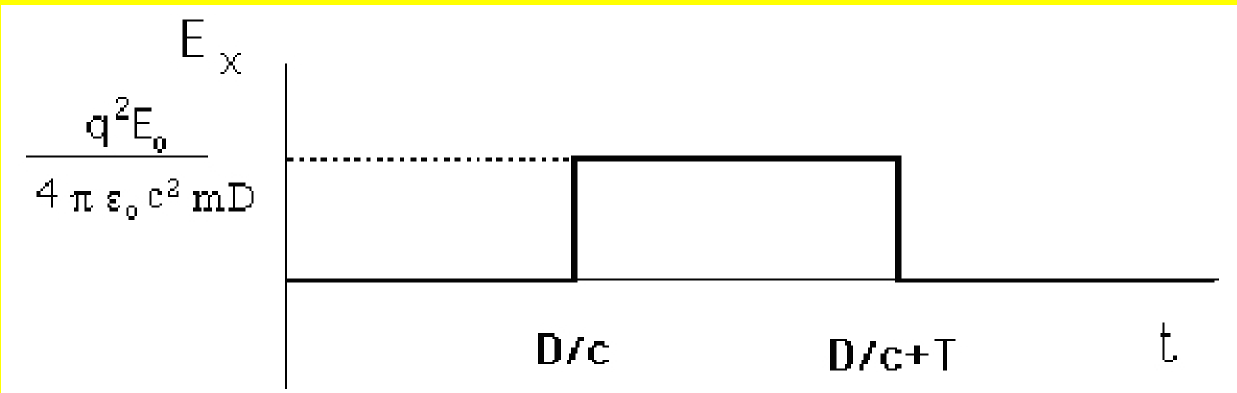
In the second way we use our formulas for the electric field of an accelerated point charge in the limit that the speed is small compared to the speed of light. In this limit, looking only at the radiation term, we have

$$\mathbf{E}_{radiation}(\mathbf{r},t) = \left[\frac{q}{4\pi\,\varepsilon_o}\frac{1}{c}\frac{\hat{\mathbf{n}}\ \mathbf{x}\left\{(\hat{\mathbf{n}}-\boldsymbol{\beta})\ \mathbf{x}\ \dot{\boldsymbol{\beta}}\right\}}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3 R}\right]_{ret} = \left[\frac{q}{4\pi\,\varepsilon_o}\frac{1}{c^2}\frac{\hat{\mathbf{n}}\times(\mathbf{n}\times\mathbf{a})}{R}\right] \quad \text{for } \beta<<1$$

$$\mathbf{E}_{radiation}(\mathbf{r},t) = \left[\frac{q}{4\pi\varepsilon_o} \frac{1}{c^2} \frac{\hat{\mathbf{y}} \times \left(\hat{\mathbf{y}} \times \left[-\hat{\mathbf{x}} \frac{qE_o}{m} \right] \right)}{D} \right] \quad \text{for } \beta << 1$$

Which leads us to the same formula above.

(b) Sketch in the space below the non-zero radiation electric field component as seen by this observer, as a function of the observer's time *t*. Label your time axis and electric field component axes appropriately. When you label your electric field component axis, put your label(s) in terms of E_o, *q*, *m*, *D*, and fundamental constants.



(c) What is the total energy radiated into electric dipole radiation in this process? Give your answer in terms of *q*, *m*, E_o, V_o, and fundamental constants.

$$\frac{1}{4\pi\varepsilon_o}\frac{2}{3}\frac{q^2a^2}{c^3}T = \frac{1}{4\pi\varepsilon_o}\frac{2}{3}\frac{q^2\left[\frac{qE_o}{m}\right]^2}{c^3}\frac{2V_o}{\left[\frac{qE_o}{m}\right]} = \boxed{\frac{4}{3}\frac{1}{4\pi\varepsilon_o}\frac{q^3E_oV_o}{mc^3}}$$

(d) Let the classical radius of the electron be $r_e = \frac{1}{4\pi\varepsilon_o}\frac{q^2}{mc^2}$. What is the ratio of the total radiated energy from part (c) to the initial kinetic energy of the particle? Write this ratio in terms of r_e, *d*, V_o and *c* alone. [Hint: your answer should have no dimensions--that is, it should only involve the ratios of lengths, speeds, etc.].

$$\frac{4}{3}\frac{1}{4\pi\varepsilon_o}\frac{q^3E_oV_o}{mc^3} / \left[\frac{1}{2}mV_o^2\right] = \frac{4}{3}\frac{1}{4\pi\varepsilon_o}\frac{q^3E_oV_o}{mc^3} / \left[d\,q\,E_o\right] = \frac{4}{3}\frac{q^2}{4\pi\varepsilon_o mc^2}\frac{V_o}{d\,c} = \frac{4}{3}\frac{r_e}{d}\frac{V_o}{c}$$

Problem 2: Relativistic dynamics of a magnetic monopole

You hear on the evening news that a magnetic dipole has finally been discovered, and you quickly go to Griffith to look up the manifestly covariant equation for relativistic dynamics for magnetic dipoles, but Griffith does not have it. You sit down and try to figure it out yourself.

If the "magnetic monopole point charge" is q_m, and its mass is m_m, you intuit that ***if you have only a magnetic field***, the non-relativistic equation of motion will be $m_m\mathbf{a} = q_m\mathbf{B}$. You want to go from this to the fully relativistic equation, including the effect of electric fields.

(a) What does your guess for the 4 vector equation for the relativistic dynamics of the magnetic monopole looks like in manifestly covariant form? Let the 4 vector position of the magnetic monopole be $X^\mu(t) = \begin{pmatrix} ct \\ \mathbf{X}(t) \end{pmatrix}$, its ordinary velocity be $\mathbf{u}(t) = \dfrac{d\,\mathbf{X}(t)}{dt}$, and its 4 vector velocity be $\eta^\mu = \dfrac{d}{d\tau}X^\mu(\tau)$. Choose the constants in your manifestly covariant equation so that you recover $m_m\mathbf{a} = q_m\mathbf{B}$ in the non-relativistic case with no electric field.

From the formula sheet we have for an electric monopole that $m\dfrac{d}{d\tau}\eta^\mu = qF^{\mu\sigma}\eta_\sigma$, so we guess that for a magnetic monopole we should have

$$m_m\frac{d}{d\tau}\eta^\mu = \frac{1}{c}q_m G^{\mu\sigma}\eta_\sigma$$

where we have put in the factor of $\dfrac{1}{c}$ so that we have in the non-relativistic limit that $m_m\mathbf{a} = q_m\mathbf{B}$ (see below).

(b) What are the equations from (a) in ordinary coordinate form, that is in terms of the ordinary position and velocity of the monopole. Give both the time and space components of your 4 vector equation from (a).

$$m_m\frac{d}{d\tau}\eta^\mu = \frac{1}{c}q_m G^{\mu\sigma}\eta_\sigma \quad \text{and} \quad \eta^\mu = \frac{d}{d\tau}X^\mu(\tau) = \begin{pmatrix} \gamma_u c \\ \gamma_u\mathbf{u}(t) \end{pmatrix}$$

$$m_m\frac{d}{d\tau}\begin{pmatrix} \gamma_u c \\ \gamma_u\mathbf{u}(t) \end{pmatrix} = \frac{1}{c}q_m \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}\begin{pmatrix} -\gamma_u c \\ \gamma_u u_x(t) \\ \gamma_u u_y(t) \\ \gamma_u u_z(t) \end{pmatrix}$$

$$m_m \frac{d}{d\tau}\begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix} = q_m \begin{pmatrix} \gamma_u \mathbf{u}\cdot\mathbf{B}/c \\ \gamma_u \left[B_x - \frac{1}{c^2}\left(u_y E_z - u_z E_y\right)\right] \\ \gamma_u \left[B_y - \frac{1}{c^2}\left(u_z E_x - u_x E_z\right)\right] \\ \gamma_u \left[B_z - \frac{1}{c^2}\left(u_x E_y - u_y E_x\right)\right] \end{pmatrix} = q_m \begin{pmatrix} \gamma_u \mathbf{u}\cdot\mathbf{B}/c \\ \gamma_u \frac{1}{c^2}\left[\mathbf{B} - \mathbf{u}\times\mathbf{E}\right] \end{pmatrix}$$

So our equations for the magnetic monopole are

$$m_m \gamma_u \frac{d}{dt}\begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix} = q_m \begin{pmatrix} \gamma_u \mathbf{u}\cdot\mathbf{B}/c \\ \gamma_u \frac{1}{c^2}\left[\mathbf{B} - \mathbf{u}\times\mathbf{E}\right] \end{pmatrix}$$

Or

$$\frac{d}{dt} m_m \gamma_u c^2 = q_m \mathbf{u}\cdot\mathbf{B}$$

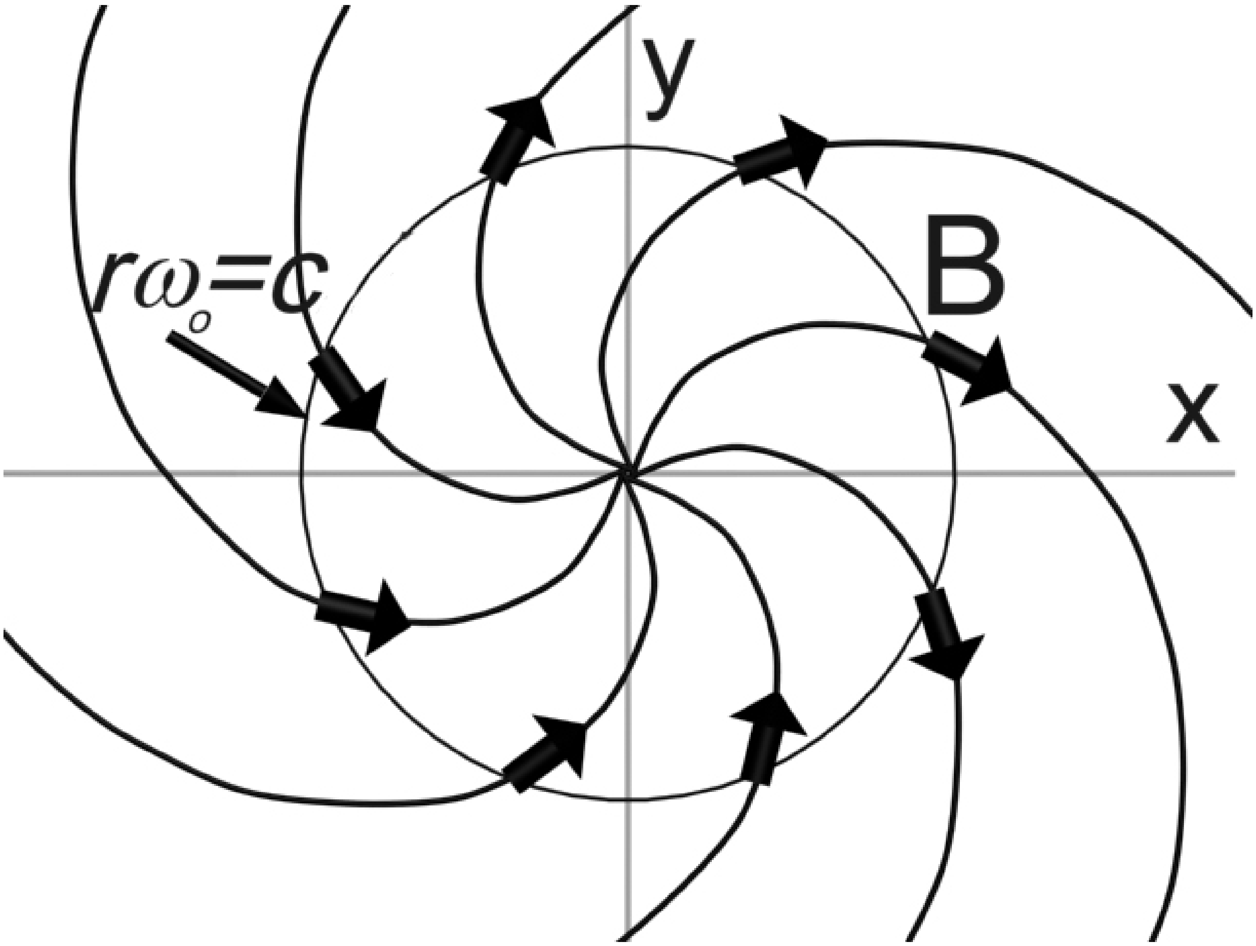
$$m_m \frac{d}{dt}\gamma_u \mathbf{u}(t) = q_m \left[\mathbf{B} - \frac{1}{c^2}\mathbf{u}\times\mathbf{E}\right]$$

Problem 3: Angular momentum loss for a rotating magnetic dipole

In Problem 4.4 of Problem Set 4, you showed that the magnetic field in the *x*-*y* plane of a magnetic dipole lying in the *x*-*y* plane and rotating about the *z*-axis was given by the following expression far from the dipole:

$$\mathbf{B}(\mathbf{r},t)\Big|_{\theta=\pi/2} = \frac{\mu_o \omega_o m_o \sin\left[\phi - \omega_o\left(t - r/c\right)\right]}{4\pi r^2 c^2}\left[c\hat{\mathbf{r}} - r\omega_o\hat{\boldsymbol{\phi}}\right]$$

These field lines at once instant of time are plotted in the figure below. We are looking down on the x-y plane from above. The circle is drawn in the *xy* plane at a value of *r* such that $r\omega_o = c$. Over one-half of the circle the magnetic field lines point outward, and over the other half of the circle the magnetic field lines point inward, as indicated on the plot by the thick arrows.



(a) In the Figure above, at the base of every thick arrow, indicate the direction of $\overleftrightarrow{\mathbf{T}}_{\mathbf{B}} \cdot \hat{\mathbf{r}}$.

Over the outward pointing half of the circle the angle between $\hat{\mathbf{r}}$ and $\mathbf{B}$ is 45 degrees, and continuing on in the same sense to 90 degrees gives us a direction perpendicular to the radius vector and in the $-\hat{\boldsymbol{\phi}}$ direction. Over the inward point half of the circle the angle between $\hat{\mathbf{r}}$ and $\mathbf{B}$ is 135 degrees, and continuing on in the same sense to 270 degrees again gives us a direction perpendicular to the radius vector and in the $-\hat{\boldsymbol{\phi}}$ direction.

(b) The vector angular momentum flux density in the $\hat{\mathbf{r}}$ direction at the base of any of the thick arrows is given by $-\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}_{\mathbf{B}} \cdot \hat{\mathbf{r}}\right)$. What is the direction of this angular momentum flux density on the half of the circle where the magnetic field $\mathbf{B}$ points inward? What is its direction on the half of the circle where $\mathbf{B}$ points outward?

In both cases the angular momentum flux density is in the $+\hat{\mathbf{z}}$ direction, as we expect.

(c) Now consider the case that r has any value, not limited to $r\omega_o = c$. Give an expression for $-\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}_{\mathbf{B}} \cdot \hat{\mathbf{r}}\right)$, using the equation for $\mathbf{B}(\mathbf{r},t)\big|_{\theta=\pi/2}$ on the previous page. How does your expression for this quantity fall with r? Is this a fall-off which will result in a net angular momentum flux to infinity? Why?

$$-\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\cdot\hat{\mathbf{r}}\right)=-\mathbf{r}\times\left(\left(\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\cdot\hat{\mathbf{r}}\right)_r\hat{\mathbf{r}}+\left(\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\cdot\hat{\mathbf{r}}\right)_\phi\hat{\boldsymbol{\phi}}\right)=-\mathbf{r}\times\left(\left(\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\cdot\hat{\mathbf{r}}\right)_\phi\hat{\boldsymbol{\phi}}\right)=-r\left(\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\cdot\hat{\mathbf{r}}\right)_\phi\hat{\mathbf{z}}$$

$$\left(\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\cdot\hat{\mathbf{r}}\right)_\phi=\left(\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\right)_{r\phi}=\mu_oB_rB_\phi=-\left[\frac{\mu_o\omega_om_o\sin\left[\phi-\omega_o\left(t-r/c\right)\right]}{4\pi r^2c^2}\right]^2 cr\omega_o$$

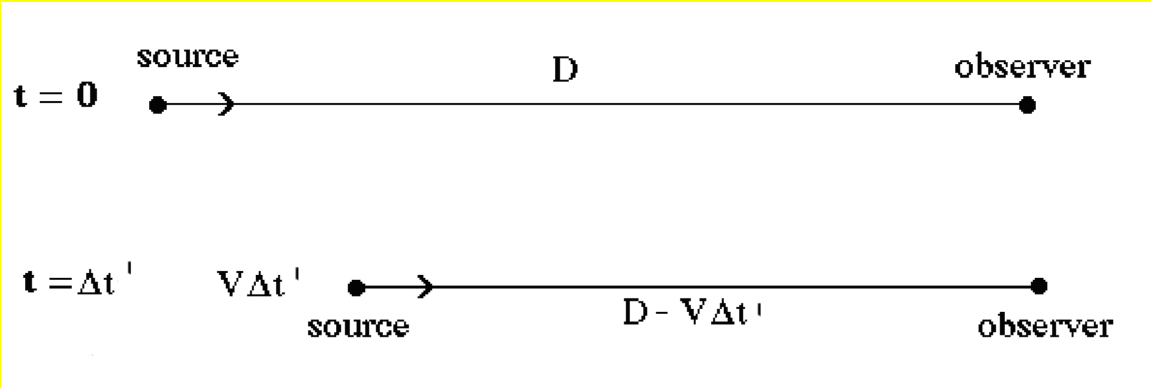
$$-\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\cdot\hat{\mathbf{r}}\right)=r\left\{\frac{\mu_o\omega_om_o\sin\left[\phi-\omega_o\left(t-r/c\right)\right]}{4\pi r^2c^2}\right\}^2 cr\omega_o\hat{\mathbf{z}}$$

This will carry of angular momentum to infinity because it goes as $\frac{1}{r^2}$ and when we multiply by $da=r^2d\Omega$ we will get a net angular momentum carried to infinity.

Problem 4: Relativistic particle motion

(a) A spaceship with speed $V<c$ moves directly toward an observer in the laboratory frame, firing laser blasts with a time separation $\Delta t'$ at the observer, where $\Delta t'$ is the time separation of the blasts as measured by our infinite grid of observers at rest in the laboratory frame. What is the time separation Δt between these laser blasts when they arrive at the observer, in terms of V, c and $\Delta t'$? Justify your answer.

Consider two laser blasts. Assume that the first blast is emitted at $t = 0$. Suppose also at time $t = 0$ that the observer and the source are separated by a distance D (see Figure)



The observer will see the first blast arrive at a time $t_1=D/c$. How about the second blast? Well, if it is emitted a time $\Delta t'$ after the first blast, and the source is assumed to be moving directly toward the observer at speed V, the source will be only a distance $D-V\Delta t'$ from the observer when the second blast is emitted. The second blast will then arrive at the observer at a time $t_2 = \Delta t' + \left(D - V\Delta t'\right)/c$ after the first blast. Since the time difference between blast as seen by the observer is $\Delta t = t_2 - t_1$, we have, as above

$$\Delta t = t_2 - t_1 = \left[\Delta t' + (D - V\Delta t')/c\right] - D/c = (1 - V/c)\Delta t'$$

We could also start with the general expression $t'_{ret} = t - \left|\mathbf{r} - \mathbf{X}(t'_{ret})\right|/c$ and differentiate with respect to t'_{ret}, giving $1 = \dfrac{dt}{dt'_{ret}} + \hat{\mathbf{n}}\cdot\boldsymbol{\beta} \Rightarrow \dfrac{dt}{dt'_{ret}} = 1 - \hat{\mathbf{n}}\cdot\boldsymbol{\beta}$. Then applying the situation we have described above, we have $\Delta t = \left(1 - V/c\right)\Delta t'_{ret}$. I prefer the first way of doing this, since the second is mathematical and hides the physics. I am not sure how much credit I will give for this way of doing it.

(b) The four acceleration vector is defined by $\Xi^\mu = \dfrac{d}{d\tau}\eta^\mu$ and is given in terms of ordinary acceleration and velocity by $\Xi^\mu = \gamma_u{}^2 \begin{pmatrix} \gamma_u{}^2 \left(\dfrac{\mathbf{u}\cdot\mathbf{a}}{c}\right) \\ \mathbf{a}(t) + \gamma_u{}^2 \mathbf{u}\left(\dfrac{\mathbf{u}\cdot\mathbf{a}}{c^2}\right) \end{pmatrix}$. From this expression for Ξ^μ, how would you compute the acceleration of the particle in its instantaneous rest frame given quantities measured in the lab frame? This is not a question about Lorentz transformations, it is a question about Lorentz invariants. You do not have to carry out the computation, just indicate how you would do it.

The Lorentz invariant length of the four acceleration is $\Xi^\mu\Xi_\mu = -\left(\Xi^0\right)^2 + \left(\Xi^1\right)^2 + \left(\Xi^2\right)^2 + \left(\Xi^3\right)^2$

We can compute this given the ordinary velocity and acceleration as measured in the laboratory frame, using the expression above. But in the instantaneous rest frame of the particle, the particle speed is zero and its acceleration is $\mathbf{a}_{\text{rest fame}}$, so the invariant length of the four acceleration vector is just $(a_{\text{rest fame}})^2$. Thus we have

$$\left(a_{rest\ frame}\right)^2 = \gamma_u{}^4 \left[-\left[\gamma_u{}^2 \left(\frac{\mathbf{u}\cdot\mathbf{a}}{c}\right)\right]^2 + \left(\mathbf{a}(t) + \gamma_u{}^2\mathbf{u}\left(\frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\right)\right)\cdot\left(\mathbf{a}(t) + \gamma_u{}^2\mathbf{u}\left(\frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\right)\right)\right]$$

You do not need to go any further than this.