DETACH THESE SHEETS & USE AT YOUR CONVENIENCE

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_o}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_o \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\int_{\substack{\text{closed}\\\text{surface}}} \mathbf{E} \cdot \hat{\mathbf{n}}\, da = \frac{q_{\text{enclosed}}}{\varepsilon_o}$$

$$\int_{\substack{\text{bounding}\\\text{contour}}} \mathbf{E} \cdot d\,\mathbf{l} = -\frac{d}{dt} \int_{\substack{\text{open}\\\text{surface}}} \mathbf{B} \cdot \hat{\mathbf{n}}\, da$$

$$\int_{\substack{\text{bounding}\\\text{contour}}} \mathbf{B} \cdot d\,\mathbf{l} = \mu_o \varepsilon_o \left[I_{\text{through}} + \frac{d}{dt} \int_{\substack{\text{open}\\\text{surface}}} \mathbf{E} \cdot \hat{\mathbf{n}}\, da \right]$$

$$\int_{\substack{\text{closed}\\\text{surface}}} \mathbf{B} \cdot \hat{\mathbf{n}}\, da = 0$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \mathbf{A} = -\mu_o \mathbf{J}$$

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \phi = -\frac{\rho}{\varepsilon_o}$$

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$$

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\, \varepsilon_o} \int \frac{\rho(\mathbf{r}', t'_{ret})}{|\mathbf{r} - \mathbf{r}'|} d^3 x'$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_o}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t'_{ret})}{|\mathbf{r} - \mathbf{r}'|} d^3 x'$$

$$t'_{ret} = t - |\mathbf{r} - \mathbf{r}'| / c$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{r^3} \frac{[3\hat{\mathbf{n}}(\mathbf{p} \cdot \hat{\mathbf{n}}) - \mathbf{p}]}{4\pi\, \varepsilon_o} + \frac{1}{c\, r^2} \frac{[3\hat{\mathbf{n}}(\dot{\mathbf{p}} \cdot \hat{\mathbf{n}}) - \dot{\mathbf{p}}]}{4\pi\, \varepsilon_o} + \frac{1}{rc^2} \frac{(\ddot{\mathbf{p}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}}{4\pi\, \varepsilon_o}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_o}{4\pi} \left[\frac{\dot{\mathbf{p}}}{r^2} + \frac{\ddot{\mathbf{p}}}{c\, r} \right] \times \hat{\mathbf{n}}$$

all evaluated at $t' = t - r / c$

$$\frac{\partial}{\partial t} \left[\frac{1}{2} \varepsilon_o E^2 + \frac{B^2}{2\mu_o} \right] + \nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_o} \right) = -\mathbf{E} \cdot \mathbf{J}$$

Energy density of field $= \frac{1}{2} \varepsilon_o E^2 + \frac{B^2}{2\mu_o}$

Energy flux density $= \frac{\mathbf{E} \times \mathbf{B}}{\mu_o}$

$$\frac{\partial}{\partial t} [\varepsilon_o \mathbf{E} \times \mathbf{B}] + \nabla \cdot \left(-\overleftrightarrow{\mathbf{T}} \right) = -[\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}]$$

Momentum density $= \varepsilon_o \mathbf{E} \times \mathbf{B}$

Momentum flux density $= -\overleftrightarrow{\mathbf{T}}$

$$\frac{\partial}{\partial t} \left[\varepsilon_o \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) \right] + \nabla \cdot \left(-\mathbf{r} \times \overleftrightarrow{\mathbf{T}} \right) = -\mathbf{r} \times [\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}]$$

Angular momentum density $= \varepsilon_o \mathbf{r} \times (\mathbf{E} \times \mathbf{B})$

Angular momentum flux density $= -\mathbf{r} \times \overleftrightarrow{\mathbf{T}}$

$$\overleftrightarrow{\mathbf{T}} = \varepsilon_o \left[\mathbf{EE} - \frac{1}{2} \overleftrightarrow{\mathbf{I}}\, E^2 \right] + \frac{1}{\mu_o} \left[\mathbf{BB} - \frac{1}{2} \overleftrightarrow{\mathbf{I}}\, B^2 \right]$$

$\overleftrightarrow{\mathbf{T}}_{\mathbf{E}} \cdot \hat{\mathbf{n}}$ lies in the plane defined by $\mathbf{E}$ and $\hat{\mathbf{n}}$.

The magnitude of $\overleftrightarrow{\mathbf{T}}_{\mathbf{E}} \cdot \hat{\mathbf{n}}$ is always $\frac{1}{2} \varepsilon_o E^2$

If you go an angle θ to get to $\mathbf{E}$ from $\hat{\mathbf{n}}$, go an angle 2θ in the same sense to get to $\overleftrightarrow{\mathbf{T}}_{\mathbf{E}} \cdot \hat{\mathbf{n}}$

All of the above apply to $\overleftrightarrow{\mathbf{T}}_{\mathbf{B}} \cdot \hat{\mathbf{n}}$ except for the fact that $\left| \overleftrightarrow{\mathbf{T}}_{\mathbf{B}} \cdot \hat{\mathbf{n}} \right|$ is always $B^2 / 2\mu_o$

$$\mathbf{p}(t') = \int \mathbf{r}'\rho(\mathbf{r}',t')d^3x'$$

$$\frac{dW}{d\Omega\ dt} = \frac{c}{\mu_o}(\mathbf{E}\times\mathbf{B})\cdot(r^2\ \hat{\mathbf{r}})$$

$$\frac{dW_{el\ dip}}{d\Omega\ dt} = \frac{c\ r^2}{\mu_o}\left[\frac{\mu_o}{4\pi}\frac{\ddot{\mathbf{p}}\,\mathbf{x}\,\hat{\mathbf{n}}}{c\ r}\right]^2 = \frac{\mu_o\ \ddot{p}^{\,2}}{(4\pi)^2 c}\sin^2\theta$$

$$\frac{dW_{el\ dip}}{dt} = \frac{1}{4\pi\,\varepsilon_o}\frac{2}{3}\frac{|\ddot{\mathbf{p}}|^2}{c^3}$$

$$\frac{dW_{el\ dip}}{dt} = \frac{1}{4\pi\,\varepsilon_o}\frac{2}{3}\frac{q^2a^2}{c^3}$$

$$\mathbf{m}(t') = \frac{1}{2}\int \mathbf{r}'\times\mathbf{J}(\mathbf{r}',t')d^3x'$$

$$\frac{dW_{mag\ dip}}{dt} = \frac{\mu_o}{4\pi}\frac{2|\ddot{\mathbf{m}}|^2}{3c^3}$$

$$\rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{X}(t))$$

$$\mathbf{J}(\mathbf{r},t) = q\mathbf{u}(t)\delta^3(\mathbf{r}-\mathbf{X}(t))$$

$$\mathbf{u}(t) = \frac{d}{dt}\mathbf{X}(t)$$

$$\mathbf{p}(t') = \int \mathbf{r}'\rho(\mathbf{r}',t')d^3x' = q\mathbf{X}(t')$$

$$\mathbf{m}(t') = \frac{1}{2}\int \mathbf{r}'\times\mathbf{J}(\mathbf{r}',t')d^3x' = \frac{q}{2}\mathbf{X}(t')\times\mathbf{u}(t')$$

For a localized distribution of current,

$$\int_{\textbf{all space}} \mathbf{J}(\mathbf{r}',t')d^3x' = -\int_{\textbf{all space}} \left(\mathbf{r}'\left[\nabla'\cdot\mathbf{J}(\mathbf{r}',t')\right]\right)d^3x'$$

$$\frac{\partial\rho(\mathbf{r}',t')}{\partial t'} + \nabla'\cdot\mathbf{J}(\mathbf{r}',t') = 0$$

$$|\,\mathbf{r}-\mathbf{r}'\,| = r - \hat{\mathbf{n}}\cdot\mathbf{r}' + ... \qquad \hat{\mathbf{n}} = \mathbf{r}/|\mathbf{r}|$$

$$\frac{1}{|\,\mathbf{r}-\mathbf{r}'\,|} = \frac{1}{r} + \frac{\hat{\mathbf{n}}\cdot\mathbf{r}'}{r^2} + ...$$

$$E_{radiation} = cB_{radiation}$$

$$\rho(\mathbf{r}',t'_{ret}) = \rho(\mathbf{r}',t - r/c + \hat{\mathbf{n}}\cdot\mathbf{r}'/c + ...) \ \text{ Taylor series}$$

$$\rho(\mathbf{r}',t'_{ret}) \cong \rho(\mathbf{r}',t - r/c) + \frac{\hat{\mathbf{n}}\cdot\mathbf{r}'}{c}\frac{\partial}{\partial t'}\rho(\mathbf{r}',t - r/c) + ...$$

Problem 1

The general solution for the potentials given the sources everywhere in space and time is

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\,\varepsilon_o}\int \frac{\rho(\mathbf{r}',t'_{ret})}{|\,\mathbf{r}-\mathbf{r}'\,|}d^3x' \quad \mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t'_{ret})}{|\,\mathbf{r}-\mathbf{r}'\,|}d^3x' \quad t'_{ret} = t - |\,\mathbf{r}-\mathbf{r}'\,|/c$$

Suppose we are far away from a localized distribution of charge and current that is varying slowly in time.

a. From the equations above, with suitable approximations, derive the *radiation part* of the magnetic field associated with the electric dipole moment $\mathbf{p}(t)$.

b. Given your answer in (a), which must be transverse to the radial direction $\hat{\mathbf{n}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$, what is the associated radiation electric field?

c. What is the rate at which energy is flowing outward per unit area in the radial direction?

Problem 2

A capacitor with capacitance C is in a circuit with an inductor with inductance L. There is no resistance as we normally think of it in this circuit. The capacitor has maximum energy and charge Q_o at $t = 0$, and the energy then sloshes back and forth between the inductor and capacitor at a frequency $\omega = 1/\sqrt{LC}$. We assume that the dimensions of this circuit are such that the speed of light transit time across the circuit is much shorter than $2\pi/\omega$. The distance between the plates of the capacitor is d, and the area of the plates is A_C, so that $C = \frac{\varepsilon_o A_C}{d}$. Take the electric dipole moment of the capacitor to be the charge times the distance between the plates. The inductance is a solenoid, with N turns, cross-sectional area A_L, and length h, so that $L = \frac{\mu_o N^2 A_L}{h}$. Assume that the dipole moment of the capacitor is Qd.

(a) What is the time-averaged rate at which this system radiates electric dipole radiation, in terms of d, ω, c, ε_o, and Q_o?

(b) Take the total energy radiated in one period of the oscillation (your answer in (a) times $\frac{2\pi}{\omega}$ and divide it by the average energy in the capacitor, $\frac{Q_o^2}{4C}$. Show this ratio is small if the speed of light transit time across the capacitor is small.

(c) The current $I(t)$ in this circuit is given by $I(t) = \frac{d}{dt}Q(t)$, so that it is clear that the time-averaged value of I^2 is $\langle I^2 \rangle = \frac{\omega^2 Q_o^2}{2}$. Use this relation to write your answer in (a) for the energy radiated as $\langle I^2 \rangle R_{radiation}$, where $R_{radiation}$ is the "radiation resistance", and has units of ohms. Give an expression for $R_{radiation}$ in terms of d, ω, c, and ε_o.

(d) Using $\omega = \frac{1}{\sqrt{LC}}$, and the equations for L and C given above, to show that the radiation resistance you have from (b) can be written in the form $c\mu_o$ times a dimensionless expression which involves the geometry of the capacitor and inductor, and N. The constant $c\mu_o$ has dimensions of ohms ($c\mu_o$ = 377 ohms), and is sometimes called the radiation resistance of free space. This "radiation resistance" has the same effect as a true resistance--the energy in the circuit slowly decreases as it is irreversibly lost to the system through radiation.